Recursion

Recall factorial: $n! \stackrel{\text{df}}{=} n \times (n-1) \times \ldots \times 2 \times 1$

Written more formally:

$$n! \stackrel{\text{df}}{=} \begin{cases} 1 & \text{if} \quad n = 0 \\ n \times (n-1)! & \text{if} \quad n > 0 \end{cases}$$

This is a *recursive* definition — $n!$ is defined in terms of $(n-1)!$

In programming we say that a function (subroutine) is recursive if, when called, it may be called again before it returns.


```
int factorial(int n)
{
  int result = 1;
  if (n > 0) {
    result = n * factorial(n-1);
  }
  return result;
}
```

Note: if `foo` calls `bar` and `bar` calls `foo` then they're both recursive.

Stack Frame/Invocation Record

When a function calls another function, the system must save:

- local variables,
- registers,
- instruction to return to

— called the *invocation record*.

This information is needed in LIFO order, so it's stored on a stack (in most programming languages).

Stack frame — the state of the stack of invocation records at a particular time. (Note: sometimes the location of the top of the stack is called the stack frame or stack frame pointer.)

Aside: Trees and Graphs

A *graph* is a set of *vertices*, V, and *edges*, E, which are pairs of verticies (i.e., $e = (v_1, v_2)$).

Two vertices are *adjacent* if there is an edge connecting them.

Two vertices are *connected* if there is a sequence of edges leading from one to the other.

A graph is *connected* if every vertex is connected to every other one.

A *cycle* is a sequence of edges leading from a vertex back to itself in which no edge appears more than once.

A *tree* is a connected graph with no cycles.

Call Trees

Illustrate the execution of an algorithm by a tree:

- each vertex represents an invocation of a function,
- each edge represents a function (the *parent*) calling another (the *child*),
- the *root* of the tree is the starting point of the algorithm (e.g., `main`) — it has no parent.

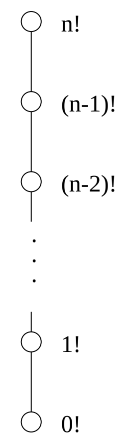
siblings are vertices with the same parent

leaf vertices have no children.

The number of verticies on the longest path from the root to a leaf is the *height* of the tree.

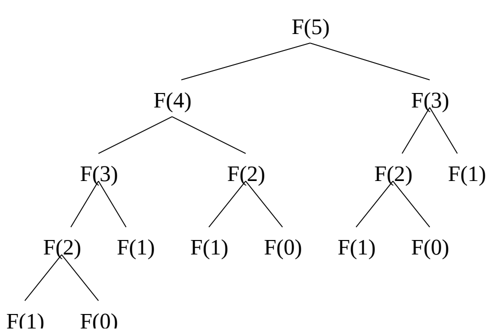
The *depth* of a vertex is the number of branches on a path from the root to the vertex.

Factorial Call Tree



Fibonacci Numbers

$$F(n) \stackrel{\text{df}}{=} \begin{cases} 0 & \text{if} \quad n = 0 \\ 1 & \text{if} \quad n = 1 \\ F(n-1) + F(n-2) & \text{if} \quad n > 1 \end{cases}$$



Towers of Hanoi

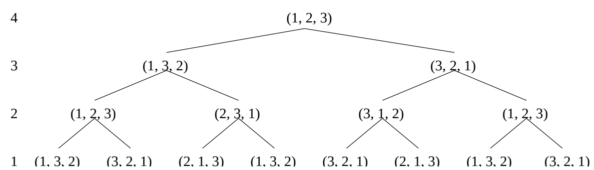
- 3 pegs, n disks, all different sizes
- Start with all disks on peg #1, ordered so that smaller disks are on top.
- Goal is to move all n disks to peg #2, subject to:
  - Move one disk at a time.
  - A larger disk can never be on top of a smaller disk.

If we can move the top $n - 1$ disks to the spare peg (#3) then we can simply move the largest disk to #2 and then move the other disks back on top of it.

Algorithm Hanoi n, $source$, $dest$, $spare$

if $n > 0$ **then**
  Move top $n-1$ disks from $source$ to $spare$ using $dest$
  Move disk n from $source$ to $dest$
  Move top $n-1$ disks from $spare$ to $dest$ using $source$
end if

Call tree



Recursion Principles

We need two things:

1) Base case — a simple instance of the problem that we know how to solve without recursion (e.g., $0!$, 1 disk Towers of Hanoi).

2) Recursive step — a means of solving a given instance of the problem by reducing it to one or more more simple instances.

   – Important that each recursive step only uses *more simple* instances.
   – A *variant expression* is a natural number expression that is smaller in recursive calls.
   – If there is a *variant expression* the recursion cannot be infinite.
   – *variant expression* $= 0$ is the base case.

See: ListR.h.

Tail Recursion

A recursive function in which the last thing it does is a recursive call to itself is *tail recursive*.

Tail recursion can be converted to iteration by re-assigning the values of local variables and using a loop.

```
int factorial(int n)
{
  int result = 1;
  for (int i = n; i > 0; i--) {
    result *= i;
  }
  return result;
}
```


Backtracking

Algorithm Maze $(start,\ end)$
  mark $start$ as seen
  $done = (start == end)$
  if $\neg done \wedge forward$ is accessible and unseen **then**
    Move forward
    $done = \mathrm{Maze}(forward, end)$
    if $\neg done$ **then**
      Move backward
    end if
  end if
  if $\neg done \wedge left$ is accessible and unseen **then**
    Move left
    $done = \mathrm{Maze}(left, end)$
    if $\neg done$ **then**
      Move right
    end if
  end if

if $\neg done \land right$ is accessible and unseen **then**
  Move right
  $done = \mathrm{Maze}(left, end)$
  if $\neg done$ **then**
    Move left
  end if
end if
return $done$