Eng. 6002 Ship Structures 1
Hull Girder Response Analysis

LECTURE 6: INCLINED BENDING AND SECTION MODULUS
Overview

- Up to this point we have assumed that the ship is upright and the bending is in the vertical plane. This is referred to as vertical bending.
- If the ship is rolling, and at some healing angle, it will be also subjected to horizontal bending.
- Since the buoyancy and gravity forces are always in the vertical direction, the bending moment is no longer aligned with the midship section and centreline planes.
Vertical and Horizontal Bending

In the case where the horizontal bending is due entirely to the inclination of the vessel, we may relate the moments as follows:

\[ M_y = M \sin \theta \]
\[ M_z = M \cos \theta \]

- Where \( M_y \) and \( M_z \) are the horizontal and vertical bending moments, respectively.
Recall that the neutral axis identifies a plane along which there is neither tension nor compression. The amount of tensile or compressive stress on any fibre is directly proportional to the distance from the neutral axis to that fibre. The beam is most highly stressed at the top and bottom, and the neutral axis is located at the centroid of the beam.
The flexure formula

- The tensile and compressive stresses in a beam are given by

\[ \sigma_y = \frac{M_y}{I} \]

where \( \sigma_y \) = the stress at point \( y \)
\( M = \) the bending moment
\( y = \) vertical distance from NA to point
\( I = \) moment of inertia of cross section about NA
Total Stress in the hull girder

- The stress at point \((y,z)\) is therefore given by:

\[
\sigma = \sigma_v + \sigma_H = \frac{M \cos \theta}{I_{NA}} y + \frac{M \sin \theta}{I_{CL}} z
\]

where \(I_{CL} = \) moment of inertia about the centreline

- When we have bending moments in both directions there will be a line of zero axial stress that we can call the **heeled** neutral axis
- To find this axis we solve for the location of zero stress
Total Stress in the hull girder

- Which leads to

\[ 0 = \sigma_v + \sigma_H = \frac{M \cos \theta}{I_{NA}} y + \frac{M \sin \theta}{I_{CL}} z \]

so

\[ \frac{\cos \theta}{I_{NA}} y = \frac{\sin \theta}{I_{CL}} z \]

or

\[ y = -\frac{I_{NA}}{I_{CL}} \tan \theta z \]
Total Stress in the hull girder

- Now we define

\[ \tan \psi = \frac{I_{NA}}{I_{CL}} \tan \theta \]

- And we have

\[ y = \tan \psi \cdot z \]
Total Stress in the hull girder

- The angle $\psi$ is the angle of the heeled neutral axis from the y-axis
Peak Stresses

- The greatest and least stresses will be associated with the maximum and minimum values of \(y\) and \(z\) (i.e. at \(z = B/2\) and \(y = y_{\text{deck}}\)).
- So we can find the angle of heal corresponding to the worst condition by setting \(d\sigma/d\theta = 0\)

\[
\frac{d\sigma}{d\theta} = -\frac{M \cos \theta}{I_{\text{NA}}} y + \frac{M \sin \theta}{I_{\text{CL}}} z = 0
\]

or

\[
\tan \theta = \frac{z}{y} \frac{I_{\text{NA}}}{I_{\text{CL}}} = \frac{Z_{\text{NA}}}{Z_{\text{CL}}}
\]
The ratio typically has a value of 0.5, which leads to a critical heel angle of about 30 degrees.

\[ \frac{d\sigma}{d\theta} = - \frac{M \cos \theta}{I_{NA}} y + \frac{M \sin \theta}{I_{CL}} z = 0 \]

or

\[ \tan \theta = \frac{z}{y} \frac{I_{NA}}{I_{CL}} = \frac{Z_{NA}}{Z_{CL}} \]
The section modulus

- Ships are largely built of plates, and the calculation of the section modulus requires determining the moments of inertia for a collection of rectangular plates.
The section modulus

- We need the parallel axis theorem to determine the moment of inertia about the neutral axis

\[ I = I_{zz} - Ah_{NA}^2 \]

where \( I = \) moment of inertia about the neutral axis
\( I_{zz} = \) moment of inertia about some axis \( zz \)
\( A = \) total area
\( h_{NA} = \) distance from \( zz \) to neutral axis
The section modulus

- The height of the neutral axis is given by

\[ h_{NA} = \frac{\sum a_i h_i}{\sum a_i} \]

This gives;

\[ h_{NA} = \frac{\sum a_i h_i}{\sum a_i} = (a_1 h_1 + a_2 h_2)/(a_1+a_2) \]
Finally, the moment of inertia calculation is summarized by the following formulas:

\[ I_{zz} = \sum a_i h_i^2 + \sum Ina_i \]

\[ I_{NA} = I_{zz} - A h_{NA}^2 \]

\[ or \]

\[ I_{NA} = \sum (Ina_i + a_i (h_i - h_{NA})^2) \]
The section modulus

- This entire calculation is often performed with a spreadsheet

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\[
A = \sum a \\
h_{NA} = \frac{\sum ah}{A} \\
I_{NA} = I_{zz} - Ah_{NA}^2
\]
In general, the two quantities to be calculated are the position of the neutral axis of the section and the moment of inertia about the neutral axis.

The choice of the axis $zz$ is arbitrary, but it makes sense to use the keel, since the height of the neutral axis is generally quoted as the height above the keel.

Most of the items in the spreadsheet can be treated as rectangles (so $I = \frac{1}{12}bt^3$).

The calculation is usually carried out for one side of the ship only, so results have to be multiplied by 2.
LECTURE 7: REVIEW OF BEAM THEORY
Overview

• In this lecture we will review the elastic response of beams and the relationship between load, shear, bending, and deflection.
• We will use the following sign conventions:
We look at a small section of the beam as a free body in equilibrium. For any static equilibrium problem, all forces and moments in all directions must be equal to zero.

\[ \sum F = 0 \text{ and } \sum M = 0 \]

Summing vertical forces we have

\[ Q(x) - (Q(x) + dQ) + P(x)dx = 0 \]

\[ \Rightarrow dQ = P(x)dx \]

so,

\[ P(x) = \frac{dQ}{dx} \]
Static Equilibrium Equations

- This means that a line load on a beam is equal to the rate of change of shear.

- Now, summing moments:

\[-M(x) - Q(x)dx - P(x)dx \frac{dx}{2} + (M(x) + dM) = 0\]

\[\Rightarrow -Q(x)dx - P(x)dx \frac{dx^2}{2} + dM = 0\]

\[dx^2\] is vanishingly small, and we can say

\[Q(x) = \frac{dM}{dx}\]
This means that shear load in a beam is equal to the rate of change of bending moment.

We can write these equations in integral form:

\[ Q(x) = \int P(x) \, dx \]

\[ = Q_o + \int_0^x P(x) \, dx \]

In words – shear is the sum of all loads from start to \( x \)

\[ M(x) = \int Q(x) \, dx \]

\[ = M_o + \int_0^x Q(x) \, dx \]
A beam is loaded and supported as shown.

a) **Draw complete shear force and bending moment diagrams.**

b) Determine the equations for the shear force and the bending moment as functions of $x$. 

Source: http://www.public.iastate.edu
Shear and Moment Diagrams example Part A: Diagrams by Inspection

- A beam is loaded and supported as shown.
  - Draw complete shear force and bending moment diagrams.
  - Determine the equations for the shear force and the bending moment as functions of $x$. 

Source: http://www.public.iastate.edu
Shear and Moment Diagrams example: Part A

- We start by drawing a free-body diagram of the beam and determining the support reactions.
- Summing moments about the left end of the beam
- Summing vertical forces

\[ \sum M_A = 7R_c - 2[4 \cdot 10] - 4(16) - 9(19) = 0 \]

\[ \Rightarrow R_c = 45 \, kN \]

\[ \sum F = R_A + R_C - (4 \cdot 10) - 16 - 19 = 0 \]

\[ \Rightarrow R_A = 30 \, kN \]
Sometimes knowing the maximum and minimum values of the shear and bending moment are enough.

So we will determine shear force diagrams by inspection.

Source: http://www.public.iastate.edu
Shear and Moment Diagrams example: Part A

- The 30-kN support reaction at the left end of the beam causes the shear force graph to jump up 30 kN (from 0 kN to 30 kN).

Source: http://www.public.iastate.edu
The distributed load causes the shear force graph to slope downward. Since the distributed load is constant, the slope of the shear force graph is constant \((dV/dx = w = \text{constant})\).

The total change in the shear force graph between points \(A\) and \(B\) is 40 kN (equal to the area under the distributed load between points \(A\) and \(B\)) from +30 kN to -10 kN.

Source: http://www.public.iastate.edu
Shear and Moment Diagrams example: Part A

- Where does the shear force become zero?
  - How much of the distributed load will it take to cause a change of 30 kN (from +30 kN to 0 kN)? Since the distributed load is uniform, the area (change in shear force) is just $10 \times b = 30$, which gives $b = 3 \text{ m}$.
  - The shear force graph becomes zero at $x = 3 \text{ m}$ (3 m from the beginning of the uniform distributed load).

Source: http://www.public.iastate.edu
Shear and Moment Diagrams example:

Part A

- The 16-kN force at $B$ causes the shear force graph to drop by 16 kN (from -10 kN to -26 kN).

Source: http://www.public.iastate.edu
Since there are no loads between points $B$ and $C$, the shear force graph is constant (the slope $dV/dx = w = 0$) at -26 kN.

Source: http://www.public.iastate.edu
The 45-kN force at C causes the shear force graph to jump up by 45 kN, from -26 kN to +19 kN.

Source: http://www.public.iastate.edu
Shear and Moment Diagrams example:

**Part A**

- There are no loads between points $C$ and $D$, the shear force graph is constant (the slope $dV/dx = w = 0$) at $+19 \text{kN}$. 

Source: http://www.public.iastate.edu
Shear and Moment Diagrams example: Part A

- Now we look at the moment diagram.
- Note that since there are no concentrated moments acting on the beam, the bending moment diagram will be continuous (i.e. no jumps).
Shear and Moment Diagrams example: Part A

- The bending moment graph starts out at zero and with a large positive slope (since the shear force starts out with a large positive value and \( \frac{dM}{dx} = V \)).

- As the shear force decreases, so does the slope of the bending moment graph. At \( x = 3 \) m the shear force becomes zero and the bending moment is at a local maximum (\( \frac{dM}{dx} = V = 0 \)).

- For values of \( x \) greater than 3 m (\( 3 < x < 4 \) m) the shear force is negative and the bending moment decreases (\( \frac{dM}{dx} = V < 0 \)).
The shear force graph is linear so the bending moment graph is a parabola.
The change in the bending moment between $x = 0$ m and $x = 3$ m is equal to the area under the shear graph between those two points.
So the value of the bending moment at $x = 3$ m is $M = 0 + 45 = 45$ kN·m.
The change in the bending moment between $x = 3$ and $x = 4$ m is also equal to the area under the shear graph.
So the value of the bending moment at $x = 4$ m is $M = 45 - 5 = 40$ kN·m.

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The bending moment graph is continuous at \( x = 4 \) m, but the jump in the shear force at \( x = 4 \) m causes the slope of the bending moment to change suddenly from \( \frac{dM}{dx} = V = -10 \) kN·m/m to \( \frac{dM}{dx} = -26 \) kN·m/m.

Since the shear force graph is constant between \( x = 4 \) m and \( x = 7 \) m, the bending moment graph has a constant slope between \( x = 4 \) m and \( x = 7 \) m (\( \frac{dM}{dx} = V = -26 \) kN·m/m).

The change in the bending moment between \( x = 4 \) m and \( x = 7 \) m is equal to the area under the shear graph between those two points.

The area of the rectangle is just \(-26 \times 3 = -78\) kN·m. So the value of the bending moment at \( x = 7 \) m is \( M = 40 - 78 = -38 \) kN·m.
Again the bending moment graph is continuous at $x = 7$ m. The jump in the shear force at $x = 7$ m causes the slope of the bending moment to change suddenly from $\frac{dM}{dx} = V = -26$ kN·m/m to $\frac{dM}{dx} = +19$ kN·m/m.

Since the shear force graph is constant between $x = 7$ m and $x = 9$ m, the bending moment graph has a constant slope between $x = 7$ m and $x = 9$ m ($\frac{dM}{dx} = V = +19$ kN·m/m).

The change in the bending moment between $x = 7$ m and $x = 9$ m is equal to the area under the shear graph between those two points. The area of the rectangle is just $M = (+19 \times 2) = +38$ kN·m. So the value of the bending moment at $x = 9$ m is $M = -38 + 38 = 0$ kN·m.
A beam is loaded and supported as shown.

a) Draw complete shear force and bending moment diagrams.

b) **Determine the equations for the shear force and the bending moment as functions of x.**
Shear and Moment Diagrams example: Part B: determining the equations

- We need to integrate the equation for the bending moment to determine the shape of beam and how much the beam will bend as a result of the loads.
- The best way to get these equations as functions of $x$ is to use equilibrium

Source: http://www.public.iastate.edu
Shear and Moment Diagrams example: Part B: determining the equations

- A free-body diagram of the left end of the beam to an arbitrary position, \(0 \, \text{m} < x < 4 \, \text{m}\) is shown.
- The right-hand portion of the beam that has been discarded exerts a shear force and a bending moment on the left-hand portion of the beam as shown.

Source: http://www.public.iastate.edu
Shear and Moment Diagrams example: Part B: determining the equations

- Summing forces in the vertical direction

\[ \sum F = 30 - (10x) - V = 0 \]

\[ \Rightarrow V = 30 - (10x) \quad (0 < x < 4) \]

- Summing moments about the right side

\[ \sum M_{rhs} = M + (10x)(x/2) - 30x = 0 \]

\[ \Rightarrow M = 30x - \left(5x^2\right) \quad (0 < x < 4) \]
Shear and Moment Diagrams example: Part B: determining the equations

- Moving along the beam to an arbitrary position, $4 \text{ m} < x < 7 \text{ m}$ and summing forces in the vertical direction

$$\sum F = 30 - (10 \cdot 4) - 16 - V = 0$$

$$\Rightarrow V = -26 \ (4 < x < 7)$$

- Summing moments about the right side

$$\sum M_{rhs} = M + (10 \cdot 4)(x - 2) + 16(x - 4) - 30x = 0$$

$$\Rightarrow M = 144 - 26x \ (4 < x < 7)$$
Shear and Moment Diagrams example: Part B: determining the equations

- Moving along the beam to an arbitrary position, $7 \text{ m} < x < 9 \text{ m}$ and summing forces in the vertical direction

\[ \sum F = 30 - (10 \cdot 4) - 16 - 45 - V = 0 \]
\[ \Rightarrow V = 19 \quad (7 < x < 9) \]

- Summing moments about the right side

\[ \sum M_{rhs} = M + (10 \cdot 4)(x - 2) + 16(x - 4) - 30x - 45(x - 7) = 0 \]
\[ \Rightarrow M = 171 - 19x \quad (7 < x < 9) \]
Verify that these equations have the appropriate character to match the shear force and bending moment diagrams developed in the first part of this problem. Also, show that these equations match the previous graphs at the points $x = 0$ m, $x = 3$ m, $x = 4$ m, $x = 7$ m, and $x = 9$ m.

Finally, show that these equations satisfy the load-shear force-bending moment relationships: $dV/dx = w$  $dM/dx = V$