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What is This?
Coupled transverse vibration modeling of drillstrings subjected to torque and spatially varying axial load

Ahmad Ghasemloonia, D Geoff Rideout and Stephen D Butt

Abstract

Predicting and mitigating unwanted vibration of drillstrings is an important subject for oil drilling companies. Uncontrolled vibrations cause premature failure of the drillstring and associated components. The drillstring is a long slender structure that vibrates in three primary coupled modes: torsional, axial and transverse. Among these coupled modes, the transverse mode is the major cause of drillstring failures and wellbore washout. Modal analysis of drillstrings reveals critical frequencies and helps drillers to avoid running the bit near critical modes. In this article, the coupled orthogonal modes of transverse vibration of a drillstring in the presence of torque and spatially varying axial force (due to mud hydrostatic effect, self-weight and hook load) are derived and the mode shapes and natural frequencies are determined through the expanded Galerkin method. The results are verified by the nonlinear finite element method. Modal mass participation factor, which represents how strongly a specific mode contributes to the motion in a certain direction, is used to determine the appropriate number of modes to retain so that computational efficiency can be maximized.

Keywords

Drillstring, transverse vibration, coupled modes, Galerkin's method, mode participation factor, finite element method, modal mass participation factor

Introduction

The subject of drillstring vibration is an ongoing challenging for drillers in oil fields. The effects of vibration on drilling performance, wellbore stability, joint failures, fatigue, etc. have led drilling companies to strengthen components or try to control and mitigate these effects to attain higher performance. In order to control or mitigate the vibration, its behavior and characteristics should be revealed and modeled analytically,1,2 experimentally in laboratory scale,3 or through field verification.4

A drillstring is a slender structure which consists of drill pipes at the upper sections and drill collars and stabilizers at the bottom sections. The drill pipes are hollow pipes (assumed 120 mm outer diameter, 10 mm thickness in this article) that are lighter than the collars (normally with an outer diameter of 120–240 mm and thickness of 30–80 mm). The bit is attached to the bottom of the collars. The lower section is called the ‘bottom hole assembly’ or BHA. The lower, heavier BHA is more easily excited to vibrate than the pipe section. This is due to the presence of an axial load, which is varying spatially along the drillstring length. This linearly varying axial load is a result of three interacting axial forces: hook load, self-weight of the drillstring and mud hydrostatic force. Drilling performance is very sensitive to the axial force at the bit, i.e. weight on bit (WOB), and WOB is one of the main parameters adjusted during drilling to improve penetration rate. Rate of penetration, or ROP, is the conventional index for measuring the efficiency of the drilling process.

The dominant role of BHA vibrations on the total drillstring vibration was verified by Dareing,5 who showed that the collars are easily excited in the lower modes. The pipes, in tension, vibrate at higher excitation frequencies,6 as will be shown in the later section on finite element modeling. In most cases, the rotational speed at the normal operational conditions is not high enough to excite the higher modes. The other reason for analyzing the BHA is that measurement-while-drilling (MWD) tools are mostly located near the bit in the BHA and any type of BHA vibrations interfere with the interpretation of...
down-the-hole (DTH) status at the surface. Finally, the BHA is composed of collars which are heavy and stiff and any type of unwanted vibration dissipates a portion of energy which is supposed to be delivered to the bit. Therefore, an increased understanding of the BHA vibrations will give valuable insight into potential vibration, and ways to avoid it, under normal operating conditions.

The primary modes of drillstring vibrations are axial, transverse and torsional. These modes are coupled together via terms containing variables like torque. Coupled torsional-bending, coupled axial-bending and coupled axial-torsional are three common combinations of coupled modes. Stick-slip oscillations are torsional, while whirling and bit bounce are examples of lateral and axial vibrations respectively. These vibrations can be transient or steady, depending on the drilling parameters such as WOB, torque, rotational speed and many other drilling conditions.

Among these primary modes, the transverse mode is said to be responsible for 75% of failures. Bending waves are not propagated to the surface via the drillstring as are torsional and longitudinal waves, due to the difference in the wave speed for different types of modes. The propagation speed for axial and torsional motions is quite high compared to the lateral motion. Therefore, for any given length of the drillstring, axial and torsional waves travel a few wavelengths to reach to the surface, while in contrast, the lateral wave travels many wavelengths to be felt at the surface. Furthermore, transverse vibration is more highly damped than the other modes due to mud effects and wellbore contact. Therefore, there could be severe bending vibrations deep in the hole, which the surface measuring tools do not indicate. As a later finite element method (FEM) modeling will show, in the lower frequencies the collars are vibrating transversely, while the pipes do not vibrate and remain approximately undeflected. This is due to the axial load distribution along the drillstring, the collars of which are mainly under compression while the pipes are under tension. As a result of the tension, the natural frequencies of the pipe section increase, while the natural frequencies of the collar section are reduced.

**Literature review**

Due to the importance of the transverse mode, several studies have been done to understand the behavior of the drillstring in this mode. Jansen studied the contact behavior between the stabilizer and the wellbore at the point of contact, using a lumped segment approach. He noticed that gravity and lateral coupling should be taken into account for more quantitative analysis. Chen et al. studied the lateral vibration of a BHA in the presence of constant weight on bit, but neglecting the torque. Berliz et al. performed a laboratory test to study lateral vibration of the drillstring and showed that the influence of axial force is greater than that of the torque on the natural frequencies of the drillstring. However, they did not consider spatially varying axial load and coupled orthogonal lateral modes. Christoforou et al. used the Lagrangian method to derive the equations of motion and study the drillstring trajectory in the lateral mode at the contact point. They used a sine wave as the dynamic WOB, without considering the torque as the coupling term for the lateral modes. Stability of the drillstring in the lateral mode was studied by Gulyaev et al. They investigated the buckling mode of the drillstring as a function of its length for special cases which have analytical solution. They showed that the buckling mode will occur at the section with the compressive axial force (collar section). In another work by Dareing, the sensitivity of drill collar vibration to the length was studied using simple beam equations without torque and axial load. Khulief et al. used the FEM to derive the frequency and modal response of a rotating drillstring without torque. They compared the results for the full order model and developed a reduced order model for the total drillstring. The string borehole interaction was also an interesting subject in the lateral mode analysis and several studies have been carried out on this subject. Laboratory test rigs, field data and FEM models have been used extensively for verification of derived natural frequencies or time response of the BHA.

Coupled orthogonal transverse vibration of the drillstring in presence of torque and linearly varying axial force has not been previously addressed. The goals of this article are analytical and numerical modeling of bending vibration of drill collars and accurate prediction of natural frequencies. Knowledge of such frequencies and understanding of the underlying physics will help drilling companies avoid resonance and reduce drillstring failures. In contrast to previous studies, this work includes the effects of steady torque and spatially varying axial load, thereby revealing coupling between the orthogonal components of lateral vibration. The FEM method is used to validate analytical model predictions. This article also uses the concept of modal mass participation factor to determine the required number of modes to retain. Retaining unnecessary modes can increase computation burden without significantly increasing model fidelity. The natural frequencies and mode shapes calculated using the methods of this article can be exported for use in low-order modal expansion models. While not matching the fidelity of high-order FEM models, low-order models can be more computationally efficient, more easily interfaced to other subsystem models, and still useful for top-level design studies.

In the following section, analytical equations of the drillstring lateral vibration, which are coupled via torque under linearly varying axial load, will be derived. The next section derives the governing equations and applied the expanded Galerkin method to solve them. Section ‘Analytical solution of the equations’ describes the FEM application and results. Section ‘Application
of the FEM method to the transverse vibration of the drillstring discusses proper model order, and the final section states conclusions and future work.

Derivation of governing equations

Drillstrings are assumed to be beamlike structures. Due to the high slenderness ratio of the drillstring and low rotational speed, among the conventional models of beam theory (Timoshenko, Rayleigh and Euler–Bernoulli), the Euler–Bernoulli beam theory is used to model the lateral vibration of drillstrings. Since the aspect ratio is greater than 10 for the drillstring, the use of classical thin beam element (Euler–Bernoulli) is valid. Several studies have been carried out to model different types of vertical beams, e.g. Hjømisen et al., which modeled the vertical beam under the effect of gravity.

In this article, the direct Newtonian approach is used to derive the mathematical model. The rotation of the drillstring (Figure 1) and gyroscopic effect are neglected, since the string could be assumed as a low-speed rotor. An element of the beam, which is under torque and linearly varying axial load, is considered. This element is located on a portion of the drillstring (beam), which is between two stabilizers (Figure 2). Two lateral directions, namely and , are considered for extracting the equations of motion. The elements in the plane and for plane are shown in Figure 2. The following derivation refers to equations (15) to (21) which can be found in Appendix 2.

The torque is resolved along the selected element as two separate bending moments in the and directions as a result of bending curvature of the drillstring (Figure 3)

\[ \vec{T} = \vec{T}_t + \vec{T}_n \]  

The vector in the normal direction will be

\[
|\vec{T}_n| = \vec{T}_n \cdot \vec{n} = (0, 0, T) \times \frac{1}{\sqrt{(\frac{du}{dz})^2 + (\frac{dv}{dz})^2 + 1}} \left( \frac{du}{dz} \frac{dv}{dz} \right)
\]

which acts as a bending moment in the and directions. The corresponding coupled term in the equation of motion will be third order. The corresponding component of , as a bending moment in the direction is \(-\frac{T(0, 0, T)}{(\frac{du}{dz})^2 + (\frac{dv}{dz})^2 + 1} \left( \frac{du}{dz} \frac{dv}{dz} \right)\). According to equation (16) and substituting for \(M_t\) (neglecting the second order differential terms)

\[
\frac{\partial}{\partial z} \left( -T \frac{dv}{dz} \right) = S_u
\]  

As it is clear from equations (18) and (19), the terms related to \(S_u\) will appear in the final equation.

![Figure 1. Schematic diagram of the drillstring in the wellbore.](Image)
of motion as \(-\frac{\partial S}{\partial z}\). Therefore, equation (20) is modified by substituting \(-\frac{\partial S}{\partial z}\) and assuming the torque is constant

\[-\frac{\partial S}{\partial z} = -\frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \left( -T \frac{\partial v}{\partial z} \right) \right) = T \frac{\partial^3 v(z, t)}{\partial z^3} \]  

(5)

Therefore, the \(T\) term will be added to the equation of motion in the \(u\) direction as a third-order derivative of \(v\). The above procedure can be repeated in the \(v\) direction with the similar results, except a negative sign due to the opposite direction.

Adding these two coupling effects to equations (20) and (21), and assuming the spatially varying axial force as \(F_{\text{axial}} = F_0 - \rho Az\) (Figure 4), the final form of the equations of motion for the two lateral directions is

\[E.l_i \frac{\partial^3 u(z, t)}{\partial z^3} - \rho A \frac{\partial^2 u(z, t)}{\partial z^2} + (F_0 - \rho Agz) \frac{\partial u(z, t)}{\partial z} + T \frac{\partial^3 v(z, t)}{\partial z^3} + \rho A \frac{\partial^2 v(z, t)}{\partial z^2} = 0 \]  

(7)

where \(F_0\) is the amount of compressive axial force at the top point of the collar section. The concentrated load at this point is due to mismatch of projected areas and the resulting resolved hydrostatic axial forces.\(^{28}\)

The derivation of the spatially varying axial force along the collar section can be found in Appendix 3.

The above set of equations is coupled by order of 3 through the torque. Although it is a set of linear equations, the varying axial load means there is not a closed form solution. In the following section, the above equations will be solved to find the natural frequencies of the drillstring as well as mode participation factors (equation (7)) for the first five modes of lateral vibration.

**Analytical solution of the equations**

In this section the expanded Galerkin method is used to derive the characteristics of the lateral vibration of the drillstring. Transforming the set of coupled PDEs to a set of linear differential ODEs is the major benefit of this method. Due to the discretized mode shape functions in this method, the results can be analyzed separately in each mode and the predominance of the modes for different sets of initial conditions can be verified. To apply this method to the above problem, \(u(z, t)\) and \(v(z, t)\) should be assumed as

\[u(z, t) = \sum_{i=1}^{m} \phi_i(z) \cdot p_i(t) \]  

(7)

\[v(z, t) = \sum_{j=1}^{m} \psi_j(z) \cdot q_j(t) \]
problem five modes will be retained. Further discussion on the number of mode shapes that should be retained can be found in the section ‘Application of the FEM method to the transverse vibration of the drillstring’. The comparison functions based on assumption of simply supported BCs\(^3,5,9,11,12\) are

\[
\phi(z) = \sin\left(\frac{\pi z}{l}\right)
\]

\[
\psi_j(z) = \sin\left(\frac{\pi z}{l}\right)
\]

If the above assumptions are substituted in equation (7) and finally in equation (6), the result is

\[
\rho A \left\{ \sin\left(\frac{\pi z}{l}\right) \dot{z}_1(t) + \sin\left(\frac{2\pi z}{l}\right) \dot{z}_2(t) + \sin\left(\frac{3\pi z}{l}\right) \dot{z}_3(t) + \sin\left(\frac{4\pi z}{l}\right) \dot{z}_4(t) + \sin\left(\frac{5\pi z}{l}\right) \dot{z}_5(t) \right\} + E I \left\{ \frac{\pi^2 \sin\left(\frac{\pi z}{l}\right)}{l^2} p_1(t) + \frac{16\pi^4 \sin\left(\frac{2\pi z}{l}\right)}{l^4} p_2(t) + \frac{81\pi^4 \sin\left(\frac{2\pi z}{l}\right)}{l^4} p_3(t) + \frac{25\pi^4 \sin\left(\frac{4\pi z}{l}\right)}{l^4} p_4(t) + \frac{625\pi^4 \sin\left(\frac{5\pi z}{l}\right)}{l^4} p_5(t) \right\}
\]

\[
+ T \left\{ - \frac{\pi^3 \cos\left(\frac{\pi z}{l}\right)}{l^3} q_1(t) - \frac{8\pi^3 \cos\left(\frac{2\pi z}{l}\right)}{l^3} q_2(t) - \frac{27\pi^3 \cos\left(\frac{3\pi z}{l}\right)}{l^3} q_3(t) - \frac{64\pi^3 \cos\left(\frac{4\pi z}{l}\right)}{l^3} q_4(t) - \frac{125\pi^3 \cos\left(\frac{5\pi z}{l}\right)}{l^3} q_5(t) \right\}
\]

\[
- \rho A g \left\{ \frac{\pi \cos\left(\frac{\pi z}{l}\right)}{l^2} p_1(t) + \frac{2\pi \cos\left(\frac{\pi z}{l}\right)}{l^2} p_2(t) + \frac{3\pi \cos\left(\frac{\pi z}{l}\right)}{l^2} p_3(t) + \frac{4\pi \cos\left(\frac{\pi z}{l}\right)}{l^2} p_4(t) + \frac{5\pi \cos\left(\frac{\pi z}{l}\right)}{l^2} p_5(t) \right\}
\]

\[
+ (F_0 - \rho A g z) \cdot \left\{ - \frac{\pi^2 \sin\left(\frac{\pi z}{l}\right)}{l^3} p_1(t) - \frac{4\pi^3 \sin\left(\frac{2\pi z}{l}\right)}{l^4} p_2(t) + \frac{9\pi^2 \sin\left(\frac{3\pi z}{l}\right)}{l^4} p_3(t) - \frac{16\pi^3 \sin\left(\frac{4\pi z}{l}\right)}{l^4} p_4(t) + \frac{25\pi^4 \sin\left(\frac{5\pi z}{l}\right)}{l^4} p_5(t) \right\} = 0
\]

(9)
For the \( v \) direction the result is as follows

\[
\rho A \left\{ \sin \left( \frac{\pi z}{l} \right) \ddot{q}_1(t) + \sin \left( \frac{2\pi z}{l} \right) \ddot{q}_2(t) + \sin \left( \frac{3\pi z}{l} \right) \ddot{q}_3(t) + \sin \left( \frac{4\pi z}{l} \right) \ddot{q}_4(t) + \sin \left( \frac{5\pi z}{l} \right) \ddot{q}_5(t) \right\} 
+ \frac{81\pi^2 \sin \left( \frac{\pi z}{l} \right)}{l} q_1(t) + \frac{256\pi^4 \sin \left( \frac{\pi z}{l} \right)}{l^2} q_3(t) + \frac{625\pi^4 \sin \left( \frac{\pi z}{l} \right)}{l^2} q_5(t) 
- T \left\{ \frac{\pi^3 \cos \left( \frac{\pi}{l} z \right) p_1(t) - 8\pi^3 \cos \left( \frac{\pi}{l} z \right) p_3(t)}{l} \right\} 
- 27\pi^2 \cos \left( \frac{\pi}{l} z \right) p_1(t) - \frac{64\pi^3 \cos \left( \frac{\pi}{l} z \right)}{l} p_4(t) 
- \frac{125\pi^3 \cos \left( \frac{\pi}{l} z \right)}{l} p_5(t) 
- \rho A g \left\{ \frac{\pi \cos \left( \frac{\pi}{l} z \right)}{l} q_1(t) + \frac{2\pi \cos \left( \frac{\pi}{l} z \right)}{l} q_3(t) + \frac{3\pi \cos \left( \frac{\pi}{l} z \right)}{l} q_1(t) + \frac{4\pi \cos \left( \frac{\pi}{l} z \right)}{l} q_4(t) + \frac{5\pi \cos \left( \frac{\pi}{l} z \right)}{l} q_5(t) \right\} 
\times \left\{ \frac{\pi^2 \sin \left( \frac{\pi}{l} z \right)}{l} q_1(t) - \frac{4\pi^2 \sin \left( \frac{\pi}{l} z \right)}{l} q_2(t) - \frac{9\pi^2 \sin \left( \frac{\pi}{l} z \right)}{l} q_3(t) - \frac{16\pi^2 \sin \left( \frac{\pi}{l} z \right)}{l} q_4(t) - \frac{25\pi^2 \sin \left( \frac{\pi}{l} z \right)}{l} q_5(t) \right\} 
= 0 \quad (10)
\]

The above equations are simplified according to the fact that the expanded Galerkin method is based on the orthogonality of modes; i.e.

\[
\int_0^l \phi_i(z) \phi_j(z) dz = \frac{l}{2} \delta_{ij} 
\int_0^l \psi_i(z) \psi_j(z) dz = \frac{l}{2} \delta_{ij} \quad (11)
\]

Since the comparison functions are assumed to be \( \phi_i(z) = \sin \left( \frac{\pi i z}{l} \right) \) and \( \psi_i(z) = \sin \left( \frac{\pi j z}{l} \right) \), applying integration by parts to equation (11) results in

\[
\int_0^l \phi_i'(z) \phi_j'(z) dz = \frac{l}{2} \left( \frac{\pi}{l} \right)^2 \delta_{ij} 
\int_0^l \psi_i'(z) \psi_j'(z) dz = \frac{l}{2} \left( \frac{\pi}{l} \right)^2 \delta_{ij} \quad (12)
\]

Using the above comparison functions, equation (9) is multiplied by \( \phi_i(z) = \sin \left( \frac{\pi i z}{l} \right) \) for \( i = 1, 2, \ldots, 5 \) and then is integrated term by term over \( z \in [0,l] \). The results for \( q_1(t) \) up to \( q_5(t) \) are shown in Figure 5 and for \( q_1(t) \) up to \( q_4(t) \) in Figure 6. Figure 7 is an expanded view of Figure 6 without \( q_5(t) \). The initial conditions were derived from the FEM solver as discussed later in the next section, to ensure geometric compatibility. Any compatible set of initial conditions will suffice, as natural frequencies are not initial condition dependent. Parameter values are shown in Table 1.

The results for \( p_1(t) \) up to \( p_5(t) \) are shown in Figure 8. There is a small variation between resonance frequencies in the vicinity of an anti-node, then the product of \( \phi_i(z) \cdot p_i(t) \) and \( \psi_i(z) \cdot q_i(t) \) determines the transverse motion of that location over the time period. The final transverse motion is the summation of these transverse motions in each mode.

In order to derive the eigenfrequencies of each mode in both directions the values of \( p(t) \) and \( q(t) \) are stored for the first five modes. If the FFT of each \( p(t) \) and \( q(t) \) is determined separately, the natural frequency will be revealed. The sampling rate was 1000, and 512 points were selected for FFT computations. The natural frequencies are compared for both directions in Figure 9. There is a small variation between resonance frequencies in the \( u \) and \( v \) directions as a result of the numerical solution. The maximum difference is 0.03 Hz.

From a practical drilling standpoint, the rotational speed should be adjusted so that it does not correspond to one of the eigenfrequencies. Effect of torque and WOB on the natural frequencies in the \( u \) and \( v \) directions has been studied as well. The torque was varied from 1 to 10 kN-m and the WOB from 30 to...
150 kN. The sensitivity of natural frequency to changes in WOB (Figure 9) is higher than the sensitivity to changes in torque (Figure 10). For the change in WOB the maximum change in the natural frequency is around 17 r/min, while for the torque the maximum change is around 4 r/min. Due to the similarity of the $u$ and $v$ frequencies, the sensitivity analysis in Figures 9 and 10 are just for the $u$ direction.

The following section applies the FEM method to the current problem and verifies the results derived by the expanded Galerkin method. Linear and nonlinear FEM has also been used to verify linear analytical results by Heisig et al.9

Figure 5. The function $p(t)$ for the first five modes (the $u$ direction).

Figure 6. The function $q(t)$ for the first five modes (the $v$ direction).
Application of the FEM method to the transverse vibration of the drillstring

The ABAQUS FEM solver package (SIMULIA Inc., version 6.7.1) was used, with Euler–Bernoulli beam elements chosen to maintain the same conditions as the mathematical model. The material specifications, given in Table 1, are the same as for the analytical model. The beam is modeled by a planar wire shape with a pipe profile to build the hollow drill collar pipe. Solution follows a three-step process with initial, general static and perturbation steps. The boundary conditions are applied at the initial step to constrain the model. The general static step is defined by the fixed

Figure 7. The function $q(t)$ for the second mode to the fifth mode.

Figure 8. The natural frequency in the $u$ and $v$ directions for the first five modes by the analytical model.
time increments with the direct full-Newtonian equation solver method to apply the static loads (body and hydrostatic forces). The resulting deformations of this step are propagated to the next step. In the last step (perturbation step) the ‘Lanczos’ eigensolver was used to extract the eigenfrequencies. This method is in contrast to the ‘subspace iteration method’ and falls into the class of transformation methods (transformation

Figure 9. The effect of WOB on the natural frequency in the u direction. WOB: weight on bit.

Figure 10. The effect of torque on the natural frequency in the u direction.
of the normalized eigenvectors through the displacement. It is widely used when the higher modes are of interest.\(^{30}\) Simply supported boundary conditions are used in the lowest node and all the DOFs except two (rotation along the beam axis and the downward motion) are constrained. The ‘cubic element’ is used, in which the shear flexibility is not considered, and this is in agreement with the Euler–Bernoulli beam theory assumption in the analytical model. Eighty elements were assumed, giving a total collar length of 200 m, and using the convergence analysis,\(^{24}\) the results did not change significantly when more elements were added.

As discussed in the introduction section, this fact that the collar section is more easily excited than the pipe section and vibrates in lower modes, while the pipes do not vibrate significantly, is verified by the FEM model as shown in Figure 11(a) to (f). Figure 11(a) and (b) are the second and the fifth modes of the drillstring, respectively, with the pipe section remaining steady. Figure 11(c), (d) and (e) are the modes 10, 25 and 50 respectively, with the vibration propagating up to the pipe section and the amplitude becoming larger in the pipe section. Figure 11(f) is the axial mode of the drillstring, which is a higher frequency mode, than the transverse vibration and the effective mass for the axial direction is a large value as associated with higher modes. The mode shapes magnitudes derived by the FEM are not absolute values, and are intended only to represent the correct shape distribution. They are shown exaggerated for clarity.

The values of frequencies for the \(u\) and \(v\) directions extracted by the FEM are shown in Figure 12 and compared with the results derived by the expanded Galerkin method. These values are in agreement with the results of the last section with slightly lower values. This is due to selection of a comparison function in the last section that is not exactly the same as the real displacement function (eigenfunction).

### Determination of appropriate model order

Another interesting result of the last section is the effective mass of each mode in any direction. The generalized mass, \(m_a\), associated with the mode \(a\) is defined as

\[
m_a = \chi_a^N M^{NM} \chi_a^M
\]

where \(M^{NM}\) is the structural matrix and \(\chi_a^N\) is the eigenvector for mode \(a\). \(M\) and \(N\) are degrees of freedom of the FEM model. In the case of eigen vector normalization, \(m_a\) is defined as unity. After finding the \(m_a\), the modal mass participation factor, \(\Gamma_{ai}\), is defined as

\[
\Gamma_{ai} = \frac{1}{m_a} \chi_a^N M^{NM} Y_i^M
\]
The modal mass participation factor indicates the participation factor of a certain motion (global translation or rotation) in the eigenvector of the mode $C_1$ in the $i$ direction. $Y_{ii}$ defines the magnitude of the rigid body response of the degree of freedom $M$ in the model. Effective masses are plotted in Figures 13 and 14.

As can be seen, for the $u$ direction, modes up to the fifth mode account almost entirely for the motion in this direction. This fact validates the truncation of modes after the fifth mode in the expanded Galerkin method. Another result is that the third mode is the dominant mode in the $u$ direction with higher effective mass and the first mode is the dominant mode in the $v$ direction, which is in agreement with the results of the analytical solutions. The same initial conditions were used in both methods. The effective mass in the axial direction happens at higher modes, compared to...
lateral modes, which is in agreement with beam theory, since the first natural frequency of the beam in the longitudinal direction is much higher than in the transverse direction. The modal mass participation factor in the $v$ direction is very similar to the $u$ direction. The difference is due to the different coupling terms in equation (6). However, the mass participation factors after the fifth mode decreases drastically in both directions. The numerical value of the modal participation factor is much higher in the axial direction with respect to the lateral motion. The approach of this section, and the ability to truncate unnecessary modes, helps to avoid excessive numerical computational costs.

Conclusions

Coupled lateral vibration of a drillstring was studied in two orthogonal transverse directions, under the action of a steady torque and varying axial load. The axial force arises from the interaction of the mud hydrostatic force, the drillstring self-weight and hook weight. As a result, a linear force profile was assumed along the drillstring. The torque along the drillstring was resolved into tangential and normal components, with the tangential component acting as the bending moment for the lateral modes. The coupled equations were derived using the direct Newtonian approach. The expanded Galerkin method was used to solve the coupled equations and reveal the natural frequencies as well as the mode participation factors. Then, the nonlinear FEM was applied to the problem with the same conditions to verify the results. The modal mass participation factor was derived for each direction and the effective number of modes for each direction was selected according to this criterion. Transverse coupled natural frequencies are more sensitive to changes in the WOB than torque. The rotary speed of the drillstring should be kept far enough from the natural frequencies to avoid excessive deflections and contact with the wellbore, both of which can cause premature failure of bottom-hole assembly components.

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### Appendix I

#### Notation

- \( A \) cross-sectional area of the collar section
- \( EI_u, EI_v \) flexural rigidity of the beam in the \( u \) and the \( v \) directions
- \( F_0 \) axial compressive force at the top point of the collar section
- \( g \) gravity acceleration
- \( m_u \) generalized mass associated with the mode \( u \)
- \( M, N \) degrees of freedom of the FEM model
- \( p_i(t) \) mode participation factor for the \( u \) direction
- \( q_i(t) \) mode participation factor for the \( v \) direction
- \( \zeta_u, \zeta_v \) shear force in the \( u \) and the \( v \) directions
- \( T^c, T^l \) torque vector in the \( z \) direction
- \( T^u, T^v \) torque components in the tangent and normal planes of the beam element
- \( u(z, t) \) displacement in first lateral direction
- \( w(z, t) \) displacement in second lateral direction
- \( \chi^N \) eigen vector of the mode \( u \)
- \( \Gamma_M \) magnitude of rigid body response of the degree of freedom \( M \) in the \( i \) direction
- \( \Gamma_{al} \) modal mass participation factor
- \( \rho \) BHA density
- \( \phi(z) \) comparison function in the \( u \) direction
- \( \psi(z) \) comparison function in the \( v \) direction
Appendix 2

The element in the \( uz \) plane is shown in Figure 2a and in the \( vz \) plane in Figure 2b. The equilibrium equations in the ‘\( uz \)’ plane are

\[
\sum M_z = 0 \Rightarrow M_z + \frac{\partial M_z}{\partial z} \text{d}z - M_v = 0 \\
+ \left( S_u \frac{\partial S_u}{\partial z} \right) \frac{\text{d}z}{2} + S_u \cdot \frac{\text{d}z}{2} = 0
\]

Neglecting the second and higher order terms of \( \text{d}z \) will result in

\[
\frac{\partial M_z}{\partial z} = S_u
\]

The force equilibrium in the ‘\( uz \)’ plane is

\[
S_u + \frac{\partial S_u}{\partial z} \text{d}z - S_u + \left( F_{\text{axial}} + \frac{\partial F_{\text{axial}}}{\partial z} \text{d}z \right) \sin \left( \theta + \frac{\partial \theta}{\partial z} \text{d}z \right) - F_{\text{axial}} \cdot \sin(\theta) + \rho A \frac{\partial^2 u(z,t)}{\partial t^2} = 0
\]

Assuming small angle \( \theta \), and discarding second or higher order terms of \( \text{d}z \), the above equation reduces to

\[
\frac{\partial S_u}{\partial z} \text{d}z + F_{\text{axial}} \frac{\partial \theta}{\partial z} \text{d}z + \frac{\partial F_{\text{axial}}}{\partial z} \text{d}z + \rho A \frac{\partial^2 u(z,t)}{\partial t^2} = 0
\]

Using the result of equation (2) in equation (4), considering that \( M_v = E.I. \frac{\partial^2 u(z,t)}{\partial z^2} \),

\[
- \frac{\partial}{\partial z} \left( \frac{\partial M_z}{\partial z} \right) = \frac{\partial}{\partial z} \left( F_{\text{axial}} \cdot \frac{\partial u(z,t)}{\partial z} \right) \\
\frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \left( E.I. \frac{\partial u(z,t)}{\partial z} \right) \right) \\
+ \frac{\partial}{\partial z} \left( F_{\text{axial}} \cdot \frac{\partial u(z,t)}{\partial z} \right) = - \rho A \frac{\partial^2 u(z,t)}{\partial t^2}
\]

Assuming constant \( E.I. \) for the drillstring in the collar section and a linearly varying axial load \( F_{\text{axial}} \)

\[
E.I. \frac{\partial^4 u(z,t)}{\partial z^4} + \frac{\partial}{\partial z} \left( F_{\text{axial}} \cdot \frac{\partial u(z,t)}{\partial z} \right) + \rho A \frac{\partial^2 u(z,t)}{\partial t^2} = 0
\]

The equation of motion in the \( v \) direction is obtained in the similar way to the \( u \) direction. Therefore

\[
E.I. \frac{\partial^4 v(z,t)}{\partial z^4} + \frac{\partial}{\partial z} \left( F_{\text{axial}} \cdot \frac{\partial v(z,t)}{\partial z} \right) + \rho A \frac{\partial^2 v(z,t)}{\partial t^2} = 0
\]

Ten coupled ordinary time differential equations in the \( u \) and \( v \) directions

\[
\pi^4 \frac{2E}{2I} \left( \frac{1}{2} - \frac{20}{9} \rho \Delta p_2 \right) - \frac{136}{225} \rho \Delta p_2 \left( \frac{1}{2} + \frac{20}{9} \rho \Delta p_2 \right) + 0.5 \rho \Delta p_2 \left( \frac{1}{2} + \frac{20}{9} \rho \Delta p_2 \right) + \frac{128\pi^2}{15} T_{q_2}^2 - \frac{\pi^2}{2} T_{q_1}^2 \\
+ \frac{2\pi^2}{4} \rho \Delta p_1 = 0
\]

\[
\pi^4 \frac{2E}{2I} \left( \frac{1}{2} - \frac{20}{9} \rho \Delta p_2 \right) - \frac{136}{225} \rho \Delta p_2 \left( \frac{1}{2} + \frac{20}{9} \rho \Delta p_2 \right) + 0.5 \rho \Delta p_2 \left( \frac{1}{2} + \frac{20}{9} \rho \Delta p_2 \right) \\
+ \frac{128\pi^2}{15} T_{q_2}^2 - \frac{\pi^2}{2} T_{q_1}^2 \\
+ \frac{2\pi^2}{4} \rho \Delta p_1 = 0
\]
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0.5 ρLΔq_d(t) − \frac{136}{225} ρgAp_1(t) + 4π^2 ρgAp_d(t)
+ \frac{1000π^2}{9r^2} Tq_3(t) = \frac{216π^2}{7r^2} Tq_3(t)
− \frac{8π^2}{15L^2} Tq_1(t) − \frac{8π^2}{L^2} Tp(t)
− \frac{8π^2}{L^2} Fq_d(t) = 0

0.5 ρLΔq_d(t) − \frac{136}{225} ρgAp_1(t) + \frac{1640}{81} ρgAp_3(t)
− \frac{1640}{81} π^2 ρgAp_4(t) − \frac{8π^2}{15L^2} Tp(t)
+ \frac{216π^2}{7r^2} Tp_1(t) + \frac{128π^4}{7r^2} \frac{EIp_1}{t}
− \frac{8π^2}{L^2} Fq_d(t) = 0

The effects of the hook load, WOB, mud hydrostatic force and self-weight are presented as a spatially varying axial force along the drillstring. The buoyant force in the drillstring should not be treated with the Archimedes’ rule and the effective tension point of view should be implemented for more precise results. Therefore, at the last point of the collar section, there are two axial upward forces, namely the WOB and the hydrostatic force at the lower cross section. The varying axial force in the collar section is

\[ F_{collar} = ρ_{collar} A_{collar} gz − WOB − F_{hydrostatic} \] (33)

Substituting for the hydrostatic force at the bottom of the collar section, the collar force will be

\[ F_{collar} = ρ_{collar} A_{collar} gz − WOB − (ρ_{mud} gf) \cdot A_{collar} \] (24)

According to dimensions and material properties given in Table 1, the varying axial force along the collar section is

\[ F_{collar} = −30 \cdot 10^3 − 2543.49 \cdot z \] (25)

The value of the axial force at the top point of the collar section (neutral point) is −30 kN, as shown in Table 1 and Figure 15.