CURVILINEAR MOTION: CYLINDRICAL COMPONENTS
(Section 12.8)

Today’s Objectives:
Students will be able to determine velocity and acceleration components using cylindrical coordinates.
APPLICATIONS

The cylindrical coordinate system is used in cases where the particle moves along a 3-D curve.

In the figure shown, the boy slides down the slide at a constant speed of 2 m/s. How fast is his elevation from the ground changing (i.e., what is \( \dot{z} \)?)
A polar coordinate system is a 2-D representation of the cylindrical coordinate system.

When the particle moves in a plane (2-D), and the radial distance, \( r \), is not constant, the polar coordinate system can be used to express the path of motion of the particle.
We can express the location of P in polar coordinates as \( r = r u_r \). Note that the radial direction, \( r \), extends outward from the fixed origin, O, and the transverse coordinate, \( \theta \), is measured counterclockwise (CCW) from the horizontal.
VELOCITY (POLAR COORDINATES)

The instantaneous velocity is defined as:

\[ v = \frac{dr}{dt} = \frac{d(ru_r)}{dt} \]

\[ v = \dot{r}u_r + r \frac{du_r}{dt} \]

Using the chain rule:

\[ \frac{du_r}{dt} = (\frac{du_r}{d\theta})(\frac{d\theta}{dt}) \]

We can prove that \( du_r/d\theta = u_\theta \) so \( du_r/dt = \dot{\theta}u_\theta \)

Therefore: \( v = \dot{r}u_r + r\dot{\theta}u_\theta \)

Thus, the velocity vector has two components: \( \dot{r} \), called the radial component, and \( r\dot{\theta} \), called the transverse component. The speed of the particle at any given instant is the sum of the squares of both components or

\[ v = \sqrt{(r \dot{\theta})^2 + (\dot{r})^2} \]
ACCELERATION (POLAR COORDINATES)

The instantaneous acceleration is defined as:

\[ a = \frac{dv}{dt} = \left( \frac{d}{dt} \right) (ru_r + r\dot{\theta}u_\theta) \]

After manipulation, the acceleration can be expressed as

\[ a = (\ddot{r} - r\dot{\theta}^2)u_r + (r\ddot{\theta} + 2r\dot{\theta})u_\theta \]

The term \((\ddot{r} - r\dot{\theta}^2)\) is the radial acceleration or \(a_r\).

The term \((r\ddot{\theta} + 2r\dot{\theta})\) is the transverse acceleration or \(a_\theta\).

The magnitude of acceleration is

\[ a = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2r\dot{\theta})^2} \]
CYLINDRICAL COORDINATES

If the particle P moves along a space curve, its position can be written as

\[ r_P = ru_r + zu_z \]

Taking time derivatives and using the chain rule:

Velocity: \[ v_p = \dot{r}u_r + r\dot{\theta}u_\theta + \dot{z}u_z \]

Acceleration: \[ a_p = (\ddot{r} - r\ddot{\theta})u_r + (r\dddot{\theta} + 2\dot{r}\dot{\theta})u_\theta + \ddot{z}u_z \]
EXAMPLE

**Given:** \( r = 5 \cos(2\theta) \) (m)
\[ \dot{\theta} = 3t^2 \text{ (rad/s)} \]
\( \theta_0 = 0 \)

**Find:** Velocity and acceleration at \( \theta = 30^\circ \).

**Plan:** Apply chain rule to determine \( \dot{r} \) and \( \ddot{r} \) and evaluate at \( \theta = 30^\circ \).

**Solution:**
\[ \theta = \int_{t_0}^{t} \dot{\theta} \, dt = \int_{0}^{t} 3t^2 \, dt = t^3 \]

At \( \theta = 30^\circ \), \( \theta = \frac{\pi}{6} = t^3 \). Therefore: \( t = 0.806 \) s.

\[ \dot{\theta} = 3t^2 = 3(0.806)^2 = 1.95 \text{ rad/s} \]
EXAMPLE (continued)

\[ \dot{\theta} = 6t = 6(0.806) = 4.836 \text{ rad/s}^2 \]

\[ r = 5 \cos(2\theta) = 5 \cos(60) = 2.5 \text{ m} \]

\[ \dot{r} = -10 \sin(2\theta)\dot{\theta} = -10 \sin(60)(1.95) = -16.88 \text{ m/s} \]

\[ \ddot{r} = -20 \cos(2\theta)\dot{\theta}^2 - 10 \sin(2\theta)\ddot{\theta} \]

\[ = -20 \cos(60)(1.95)^2 - 10 \sin(60)(4.836) = -80 \text{ m/s}^2 \]

Substitute in the equation for velocity

\[ \mathbf{v} = \dot{\mathbf{r}}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta \]

\[ \mathbf{v} = -16.88\mathbf{u}_r + 2.5(1.95)\mathbf{u}_\theta \]

\[ \mathbf{v} = \sqrt{(16.88)^2 + (4.87)^2} = 17.57 \text{ m/s} \]
EXAMPLE (continued)

Substitute in the equation for acceleration:

\[ a = (\dot{r} - r\ddot{\theta}) u_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) u_\theta \]

\[ a = [-80 - 2.5(1.95)^2] u_r + [2.5(4.836) + 2(-16.88)(1.95)] u_\theta \]

\[ a = -89.5 u_r - 53.7 u_\theta \text{ m/s}^2 \]

\[ a = \sqrt{(89.5)^2 + (53.7)^2} = 104.4 \text{ m/s}^2 \]
CONCEPT QUIZ

1. If $\mathbf{r}$ is zero for a particle, the particle is
   A) not moving.  
   B) moving in a circular path. 
   C) moving on a straight line.  
   D) moving with constant velocity.

2. If a particle moves in a circular path with constant velocity, its radial acceleration is
   A) zero.  
   B) $\ddot{r}$.  
   C) $-r\dot{\theta}^2$.  
   D) $2r\dot{\theta}$. 
GROUP PROBLEM SOLVING

**Given:** The car’s speed is constant at 1.5 m/s.

**Find:** The car’s acceleration (as a vector).

**Hint:** The tangent to the ramp at any point is at an angle

\[ \phi = \tan^{-1} \left( \frac{12}{2\pi(10)} \right) = 10.81° \]

Also, what is the relationship between \( \phi \) and \( \theta \)?

**Plan:** Use cylindrical coordinates. Since \( r \) is constant, all derivatives of \( r \) will be zero.

**Solution:** Since \( r \) is constant the velocity only has 2 components:

\[ v_\theta = r\dot{\theta} = v \cos\phi \quad \text{and} \quad v_z = \dot{z} = v \sin\phi \]
GROUP PROBLEM SOLVING (continued)

Therefore: \[ \dot{\theta} = \left( \frac{v \cos \phi}{r} \right) = 0.147 \text{ rad/s} \]

\[ \ddot{\theta} = 0 \]

\[ v_z = \dot{z} = v \sin \phi = 0.281 \text{ m/s} \]

\[ \ddot{z} = 0 \]

\[ \dot{r} = \ddot{r} = 0 \]

\[ a = (\ddot{r} - r \dot{\theta}^2)u_r + (r \ddot{\theta} + 2r \dot{\theta})u_\theta + \ddot{z}u_z \]

\[ a = (-r \dot{\theta}^2)u_r = -10(0.147)^2u_r = -0.217u_r \text{ m/s}^2 \]
ATTENTION QUIZ

1. The radial component of velocity of a particle moving in a circular path is always
   A) zero.
   B) constant.
   C) greater than its transverse component.
   D) less than its transverse component.

2. The radial component of acceleration of a particle moving in a circular path is always
   A) negative.
   B) directed toward the center of the path.
   C) perpendicular to the transverse component of acceleration.
   D) All of the above.