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Faddeev-Leverrier Method

This method is used to compute all the eigen values, vectors, inverse, etc., of any given matrix.

Let the behaviour of a physical problem be expressed in equations that can be arranged as a matrix equation $\underline{\underline{A}}\,\underline{X} = \lambda \underline{X}$ *or* $\left\lfloor \underline{\underline{A}} - \lambda \underline{\underline{I}} \right\rfloor \underline{X} = \underline{0}$ where, $\underline{\underline{A}}$, $\underline{X}$, $\underline{\underline{I}}$ & $\underline{0}$ are, a known square matrix, a column vector of variables, a unit matrix and a zero vector, respectively. The parameter λ is called the eigen value for the matrix $\underline{\underline{A}}$, while $\underline{X}$ is the corresponding eigen vector. If $\underline{\underline{A}}$ is an *n*x*n* matrix, then there will be *n* eigen values and *n* corresponding eigen vectors.

If $\left\lfloor \underline{\underline{A}} - \lambda \underline{\underline{I}} \right\rfloor \underline{X} = \underline{0}$, then the determinant $\left| \underline{\underline{A}} - \lambda \underline{\underline{I}} \right|$ must vanish. The equation $\left| \underline{\underline{A}} - \lambda \underline{\underline{I}} \right| = 0$ is a short form for a polynomial equation in the form of

$$\lambda^n + \alpha_{n-1}\lambda^{n-1} + \alpha_{n-2}\lambda^{n-2} + \alpha_{n-3}\lambda^{n-3} + \ldots\ldots\ldots\ldots + \alpha_1\lambda + \alpha_0 = 0,$$

where, α_i are the coefficients of the equation with the unknown variable λ.

Any solution (for λ) that satisfies the above polynomial gives a valid eigen value. If we substitute a valid eigen value in the equation $\left\lfloor \underline{\underline{A}} - \lambda \underline{\underline{I}} \right\rfloor \underline{X} = \underline{0}$ and solve for vector $\underline{X}$, it will be the corresponding eigen vector associated with the particular eigen value.

Faddeev-Leverrier Method

Define the trace of a matrix as the sum of the diagonal elements, $tr\left(\underline{\underline{M}}\right) = \sum_{i=1}^{n} m_{ii}$

Define a matrix $\underline{\underline{D_1}} = \underline{\underline{I}}$, where, $\underline{\underline{I}}$ is a unit matrix. $\underline{\underline{D_1}}$ and $\underline{\underline{I}}$ are the same size as the original matrix $\underline{\underline{A}}$.

Then, the polynomial coefficient $\alpha_{n-1} = -tr\left(\underline{\underline{A}}\,\underline{\underline{D_1}}\right)$

Similarly define a series of matrices, $\underline{\underline{D_i}} = \underline{\underline{A}}\,\underline{\underline{D_{i-1}}} + \alpha_{n-i+1}\,\underline{\underline{I}}$

Then, the polynomial coefficient $\alpha_{n-i} = -\frac{1}{i} tr\left(\underline{\underline{A}}\,\underline{\underline{D_i}}\right)$,   *i*=2, 3, 4, ….., *n*

This process gives all the coefficients of the polynomial equation

$$\lambda^n + \alpha_{n-1}\lambda^{n-1} + \alpha_{n-2}\lambda^{n-2} + \alpha_{n-3}\lambda^{n-3} + ... + \alpha_1\lambda + \alpha_0 = 0$$

Solve the polynomial equation to obtain the roots of the equation. These roots are the eigen values of the system. For any specific eigen value λ, the eigen vector can be obtained as a column of the matrix,

$$\underline{\underline{C}} = \sum_{i=1}^{n} \lambda^{n-i} \underline{\underline{D_i}}$$

Every column of this matrix represents the same eigen vector corresponding to the eigen value λ.

The inverse of the original matrix can be easily obtained using,

$$\underline{\underline{A^{-1}}} = -\frac{1}{\alpha_0} \underline{\underline{D_n}}$$

In solving for the eigen values as roots of the polynomial equation, the following theorems might be useful.

1. An n^{th} degree polynomial has exactly *n* roots some or all of which may be repeated. At least one will be a real root if *n* is odd. Complex roots always occur in conjugate pairs.

2. Descarte's rule of signs: The number of positive real roots is equal to (or less by an even number) the total number of sign changes in the coefficients. The number of negative real roots can be estimated by substituting -λ in place of λ.

   Ex: The equation $\lambda^2 - 4\lambda + 3 = 0$ has two sign changes for the coefficients (positive to negative and positive again). Therefore, there are either two or zero positive real roots. Similarly, substituting -λ in place of λ, $(-\lambda)^2 - 4(-\lambda) + 3 = \lambda^2 + 4\lambda + 3 = 0$ has no sign changes. Hence there are no negative real roots.

3. Gershgorin's Theorem 1:

   Eigen values of any matrix $\underline{\underline{A}} = \lfloor a_{ij} \rfloor_{nxn}$ must lie in the union of circles with centers a_{ii} and radius $r_i = -|a_{ii}| + \sum_{j=1}^{n} |a_{ij}|$.

4. Gershgorin's Theorem 2:

   If *k* circles of these do not touch the remaining *n*-*k* circles, then exactly *k* eigen vales lie in the union of the *k* circles.

Eg: For the matrix $\underline{A} = \begin{bmatrix} 4 & -2 & 1 \\ -1 & -3 & 1 \\ -1 & 1 & -4 \end{bmatrix}$, there are three eigen values. The Eigen equation $\left|\underline{\underline{A}} - \lambda \underline{\underline{I}}\right| = 0$ gives, $\lambda^3 + 3\lambda^2 - 18\lambda - 50 = 0$. This equation has one sign change of the coefficients. Therefore, there is one positive root (or eigen value) for the equation. If we substitute -λ in place of λ, $(-\lambda)^3 + 3(-\lambda)^2 - 18(-\lambda) - 50 = -\lambda^3 + 3\lambda^2 + 18\lambda - 50$, the equation shows two sign changes. Therefore, there are two (or zero) negative roots or eigen values.

Using Gershgorin's theorem, for the first row,

$$r_i = -|a_{ii}| + \sum_{j=1}^{n} |a_{ij}|$$

$$r_1 = -|a_{11}| + \sum_{j=1}^{3} |a_{1j}| = -|4| + |4| + |-2| + |1| = 3$$

The first circle will be drawn with the center at 4 and radius of 3.

Similarly, the circles for the other two rows can be drawn. As can be seen, there are two groups of circles. There should be one eigen value within the circle between +1 and +7. There are two eigen values within the union or the two circles between -1 and -6.

This can be easily confirmed by solving the cubic equation.

The roots are: 4.176, -2.638, -4.539