

Review of Introduction to Eigen Values

Eigen value problems are sometimes called as characteristic value problems and are among the most important class of physical problems for engineering use. They occur in stability problems, dynamics and vibrations, large displacement analyses and several other application areas. They appear whenever a problem has valid solution only for certain specific values of input constants. Such special values of physical constants are called as characteristic values or eigen values.

Physical problem behaviour can sometimes be expressed as $\underline{A} \underline{X} = \lambda \underline{X}$ or $[\underline{A} - \lambda \underline{I}] \underline{X} = \underline{0}$ where, \underline{A} , \underline{X} , \underline{I} & $\underline{0}$ are, a known square matrix, a column vector of variables, a unit matrix and a zero vector, respectively. The parameter λ is the eigen value for the matrix \underline{A} , while \underline{X} is the corresponding eigen vector. If \underline{A} is an $n \times n$ matrix, then there will be n eigen values and n corresponding eigen vectors.

If $[\underline{A} - \lambda \underline{I}] \underline{X} = \underline{0}$, then the determinant $|\underline{A} - \lambda \underline{I}|$ must vanish. The equation $|\underline{A} - \lambda \underline{I}| = 0$ is a short form for a polynomial equation in the form of

$$\lambda^n + \alpha_{n-1} \lambda^{n-1} + \alpha_{n-2} \lambda^{n-2} + \alpha_{n-3} \lambda^{n-3} + \dots + \alpha_1 \lambda + \alpha_0 = 0, \text{ where, } \alpha_i \text{ are simple coefficients.}$$

Any solution in terms of λ for the above polynomial gives valid eigen values. If we substitute a valid eigen value in the equation $[\underline{A} - \lambda \underline{I}] \underline{X} = \underline{0}$ and solve for vector \underline{X} , it will be the corresponding eigen vector associated with the particular eigen value.

Example Let $\underline{A} = \begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix}$.

$$\text{Then, } |\underline{A} - \lambda \underline{I}| = 0 = \begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1-\lambda & -3 \\ 2 & -4-\lambda \end{vmatrix} = (1-\lambda)(-4-\lambda) - (-3)(2) = 0$$

$$\therefore \lambda^2 + 3\lambda + 2 = 0, \text{ or } \lambda = \frac{-3 \pm \sqrt{3^2 - 8}}{2} = -1, -2$$

$$\text{Eigen vector can be obtained using, } [\underline{A} - \lambda \underline{I}] \underline{X} = \underline{0} \text{ or } \begin{bmatrix} 1-\lambda & -3 \\ 2 & -4-\lambda \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Using $\lambda = -1$ and expanding the matrix equation,

$$\begin{aligned} 2x_1 - 3x_2 &= 0 \\ 2x_1 - 3x_2 &= 0 \end{aligned} \quad \therefore x_2 = \frac{2}{3}x_1 \quad \text{or} \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2/3 \end{Bmatrix}.$$

These equations are satisfied for any set of vectors $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = k \begin{Bmatrix} 1 \\ 2/3 \end{Bmatrix}$, where, k is any arbitrary constant.

Notice that the two equations that came of the original matrix equation are not independent of each other (in this case they are the same and in some other cases, they may be related in some way to each other). We can easily verify that $\underline{A} \underline{X} = \lambda \underline{X}$ for this value of λ .

Similarly, the eigen vector solution for $\lambda=-2$ is obtained by substituting it into the matrix eq.,

$$\begin{aligned} 3x_1 - 3x_2 &= 0 \\ 2x_1 - 2x_2 &= 0 \end{aligned} \quad \therefore x_2 = x_1 \quad \text{or} \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = k \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \text{ for any arbitrary constant } k.$$

In **summary**, the 2x2 matrix

$$\underline{A} = \begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix} \text{ has two eigen values } \lambda = -1, -2.$$

Eigen value $\lambda=-1$ has an eigen vector $\begin{Bmatrix} 1 \\ 2/3 \end{Bmatrix}$. Eigen value $\lambda=-2$ has an eigen vector $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

Numerical Solution for Eigen Value Problems – Power Method

The analytical solution for finding eigen values and vectors as discussed above can be theoretically used for any matrix. The method is easy for small matrices. However, for large matrices (of $n \times n$ where, n can be in the order of thousands or even millions), this system becomes cumbersome. For such matrices, there are several numerical techniques that can be used to determine eigen values and vectors approximately. One such method is the popular Power Method. It is used to determine the largest eigen value. For other eigen values, the method is used with certain modifications.

Power Method for Determining Eigen Values

Choose any arbitrary eigen vector and multiply the matrix with it. Normalize the resultant vector by dividing it with the largest numerical value within the vector. Multiply the resulting normalized vector and the original matrix. Normalize the result again. Repeat the process till convergence is achieved. The converged vector and the corresponding normalizing value represent the largest eigen value and vector pair. In effect, the Power method seeks to apply the equation $\underline{A} \underline{X}_L = \lambda \underline{X}_R$ repeatedly for an arbitrarily chosen vector until the left hand side vector \underline{X}_L is equal to the right hand side vector \underline{X}_R .

If the power method is applied on the inverse of the original matrix, the eigen vector corresponds to the smallest eigen value. The smallest eigen value is obtained as the inverse (reciprocal) of the corresponding normalizing coefficient. The method works since the eigen values of an inverse matrix are the reciprocals of the eigen values of the original matrix. The vector associated is the corresponding eigen vector (i.e., using the inverse does not affect the vector)

For intermediate eigen vectors, the power method is applied on the inverse of a shifted matrix. Let us assume that we need to find the eigen value that is between the largest and the smallest, say, an eigen value close to 's'. Subtract this value from all the diagonal values of the original matrix. The resulting matrix is called as the shifted matrix. Apply the Power Method on the inverse of the shifted matrix. The desired eigen value is obtained by adding the value of s to the inverse of the corresponding normalizing coefficient. The eigen vectors are not affected by this process.

Caution: Every eigen vector of a given matrix satisfies the eigen equation. If a guess vector satisfies the eigen equation $\underline{A} \underline{X}_L = \lambda \underline{X}_R$ such that \underline{X}_L is equal to \underline{X}_R , it may be because it is one of the eigen vectors -not necessarily the one we are looking for. Power method leads by default to the largest eigen vector. However, if by chance we use the smallest eigen vector in the equation, it satisfies the equation in the first iteration itself. In such a case, we must verify that the chosen vector is indeed that corresponding to the largest eigen value. To verify, choose a slightly different vector and apply the power method. If the resulting eigen value is moving away from the one obtained in the first trial, the initial guess vector is wrong. Choose a different trial vector and proceed with the Power method. The wrong guess vector is not a waste since it actually corresponds to a proper vector albeit one that is not the largest.