Solution to cubic equation

The general cubic equation is of the form: $z^3 + a_2 z^2 + a_1 z + a_0 = 0$

Let

$$Q = \frac{3a_1 - a_2^2}{9}, \qquad R = \frac{9a_2 a_1 - 27a_0 - 2a_2^3}{54}, \; D = Q^3 + R^2, \qquad S = \sqrt[3]{R + \sqrt{D}}, \qquad T = \sqrt[3]{R - \sqrt{D}}$$

The roots of the cubic equation are given by,

$$z_1 = -\frac{1}{3}a_2 + (S + T)$$

$$z_2 = -\frac{1}{3}a_2 - \frac{1}{2}(S + T) + \frac{1}{2}i\sqrt{3}(S - T)$$

$$z_3 = -\frac{1}{3}a_2 - \frac{1}{2}(S + T) - \frac{1}{2}i\sqrt{3}(S - T)$$

These three equations giving the three roots of the cubic equation are sometimes known as Cardano's formula. The solution to the cubic (as well as the quartic) was published by Geromino Cardano (1501-1576) in his treatise *Ars Magna*. However, Cardano was not the original discoverer of either of these results. The hint for the cubic had been provided by Niccolò Tartaglia, while the quartic had been solved by Ludovico Ferrari. However, Tartaglia himself had probably caught wind of the solution from another source. The solution was apparently first arrived at by a little-remembered professor of mathematics at the University of Bologna by the name of Scipione del Ferro (ca. 1465-1526). While del Ferro did not publish his solution, he disclosed it to his student Antonio Maria Fior (Boyer and Merzbach 1991, p. 283). This is apparently where Tartaglia learned of the solution around 1541.

The equation for z_1 in Cardano's formula does not have an i appearing in it explicitly while z_2 and z_3 do, but this does not say anything about the number of real and complex roots (since S and T are themselves, in general, complex). However, determining which roots are real and which are complex can be accomplished by noting that if the polynomial discriminant $D > 0$, one root is real and two are complex conjugates; if $D = 0$, all roots are real and at least two are equal; and if $D < 0$, all roots are real and unequal. If $D < 0$, define

$$\theta = \cos^{-1}\left(\frac{R}{\sqrt{-Q^3}}\right)$$

Then the real solutions are of the form

$$z_1 = 2\sqrt{-Q}\cos\left(\frac{\theta}{3}\right) - \frac{1}{3}a_2$$

$$z_2 = 2\sqrt{-Q}\cos\left(\frac{\theta + 2\pi}{3}\right) - \frac{1}{3}a_2$$

$$z_3 = 2\sqrt{-Q}\cos\left(\frac{\theta + 4\pi}{3}\right) - \frac{1}{3}a_2$$

Ref: World of Mathematics, Wolfram Research at: http://mathworld.wolfram.com/CubicEquation.html