Memorial University - Eng. 5434

Ordinary Differential Equations – Boundary Value Problems

Higher order differential equations are called Initial Value Problems if all the necessary boundary conditions are given at the same point, e.g.,

$$y'' = f(x, y, y'), \qquad a \le x \le b; \qquad at\ x = a, \quad y = \alpha \quad \& \quad y' = \beta$$

These problems can be solved by

- converting the higher order equation in to a set of two first order equations using a simple transformation, e.g., $z1 = y \quad \& \quad z2 = y'$, giving,

  $$z1' = z2$$
  $$z2' = f(x, z1, z2)$$
  $$at \quad x = a, \quad z1 = \alpha \quad \& \quad z2 = \beta$$

- starting the numerical solution of the two first order equations simultaneously at $x=a$ (using any acceptable method such as Runge-Kutta).

However, some higher order differential equations may not have all the necessary conditions given at the same initial point. Such equations are in general, boundary value problems. Depending on the way these boundary conditions are specified, they can be three different types.

- if the all the conditions are for the primary dependent variable, they are called as Dirichlet boundary conditions

  $$y'' = f(x, y, y'), \qquad a \le x \le b;$$
  $$at\ x = a, \quad y = \alpha$$
  $$at\ x = b, \quad y = \beta$$

- if the conditions are for the derivatives of the primary dependent variable, they are called as Neuman boundary conditions

  $$y'' = f(x, y, y'), \qquad a \le x \le b;$$
  $$at\ x = a, \quad y' = \alpha$$
  $$at\ x = b, \quad y' = \beta$$

- they can also be mixed boundary conditions

  $$y'' = f(x, y, y'), \qquad a \le x \le b;$$
  $$at\ x = a, \quad c_1 y + c_2 y' = \alpha \qquad ,$$
  $$at\ x = b, \quad c_3 y + c_4 y' = \beta$$

  where, c_1, c_2, c_3 and c_4 are constants (or simple functions of x which become constants for specified values of x).

Depending on the type of the function in $y'' = f(x, y, y')$, these differential equations can be called as either linear or nonlinear. The equation is linear if it is of the form,

$y'' = p(x)y' + q(x)y + r(x)$, where, $p(x)$, $q(x)$ and $r(x)$ are simple functions of x (or constants).

For such problems, employ shooting method (or other suitable techniques). The shooting method involves shooting the target using an assumed initial trajectory. For example, let a typical problem involve the Dirichlet boundary conditions.

$$y'' = f(x, y, y'), \qquad a \le x \le b;$$
$$at\ x = a, \quad y = \alpha$$
$$at\ x = b, \quad y = \beta$$

Use the substitution $z1 = y \quad \& \quad z2 = y'$, giving,

$$z1' = z2; \qquad z2' = f(x, z1, z2)$$
$$at \quad x = a, \quad z1 = \alpha$$
$$at \quad x = b, \quad z1 = \beta$$

Let the solution start at the initial point $x=a$. For this, we must know both $z1$ and $z2$ at this point. Since we do not know z2 at $x=a$, assume any suitable value for it, say, $z2=G_1$ (guess value 1). Using $z1=\alpha$ and $z2=G_1$ at $x=a$, start the solution. Pick any acceptable method such as Runge-Kutta and a suitable step size for Δx. Proceed till the last step. This step will yield a value of $z1$ at $x=b$. Let this value be R_1 (result 1). If this value is equal to the target, i.e., $R_1=\beta$, then the initial guess value $z2=G_1$ is correct. If it is not, assume a different guess value at the beginning, say, $z2=G_2$. Repeat the analysis. This analysis will yield a different result for $z1$ at $x=b$. Let this value be R_2 (result 2). The correct guess for the initial value of z2 can now be interpolated from the two trials. It can be summarized as,

	At *x*=*a* (guess)	**At *x*=*b* (result)**
Trial 1	$z2=G_1$ (guess 1)	$z1=R_1$ (result 1)
Trial 2	$z2=G_2$ (guess 2)	$z1=R_2$ (result 2)
Interpolate	$z2 = G_1 + \frac{G_2 - G_1}{R_2 - R_1}(\beta - R_1)$ or, $guess_3 = guess_1 + \frac{guess_2 - guess_1}{result_2 - result_1}(target - result_1)$	$z1=\beta$ (target)

If the differential equation is linear, the interpolated third guess will give the correct solution. This is so since a linear problem behaves like a straight line. If two points (trial solutions) on a straight line are known, then all other points on it can be located by simple interpolation. In fact, the third analysis need not be actually carried out, the correct result for each step can be simply interpolated using the results of the two trials. For nonlinear problems, this procedure can be used iteratively till convergence is achieved. This interpolation method, when applied to nonlinear problems is called as secant method since the projection is done through a secant line. If more than two trials are necessary for solving a nonlinear problem, the guesses for successive trials using this method can be interpolated using,

$$guess_j = guess_{j-1} + \frac{guess_{j-1} - guess_{j-2}}{result_{j-1} - result_{j-2}}\left(t\arg et - result_{j-1}\right)$$

The solution is very similar for other types of boundary conditions (Neuman or mixed).

The interpolation can also be done using the Newton's method of root finding. For this method we need to use the derivative of the dependent variable at the end of the solution. If y_n (=z1n) represents the end result of y at x=b, and dy_n/dx represents the slope at that point, the Newton interpolation gives the next guess as,

$$guess_j = guess_{j-1} + \frac{t\arg et - z1n_{j-1}}{\left(\frac{dy_n}{dx}\right)_{j-1}}$$