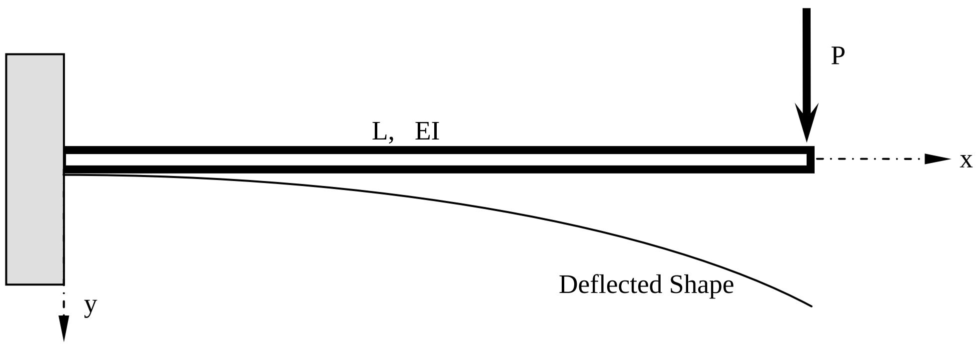
MEMORIAL UNIVERSITY OF NEWFOUNDLAND
FACULTY OF ENGINEERING AND APPLIED SCIENCE

Engineering: 5434 —– Applied Mathematical Analysis

NUMERICAL SOLUTION OF SYSTEMS OF ORDINARY DIFFERENTIAL EQ. WITH INITIAL VALUES

If systems of higher order differential equations need to be solved, convert them into a set of first order differential equations. The resultant set is solved using any of the above methods (i.e., those applicable for a single first order differential equation). The chosen method is applied to each of the functions in the set of first-order differential equations at each step in a repetitive manner.

Example: Obtain the deflections of a cantilever beam subject to a concentrated load at the end. Use, L=1.0, P/EI=0.2.



Solution: For the given beam, bending moment = $M(x) = P(L-x)$.

Deflections are governed by the differential equation $EI \dfrac{y''}{\left[1+(y')^2\right]^{3/2}} = M(x)$;

Note that the solid mechanics textbooks usually neglect the denominator term in LHS. Including the term makes the problem nonlinear and much harder to solve using classical techniques. Convert the equation into a system of first order equations by using the substitution

$y = w_1, \qquad \dfrac{dy}{dx} = w_2$. Then,

$\dfrac{dw_1}{dx} = w_2; \qquad \dfrac{dw_2}{dx} = c(L-x)(1+w_2^2)^{3/2}$, where, c=P/EI

The above are two first order differential equations that need to be solved simultaneously. The boundary conditions are,

$w_1\big|_{x=0} = 0; \quad w_2\big|_{x=0} = 0$

Both conditions are given at the same point.
Use c=0.2, h=0.1 and n=10.

Let us use Modified Euler's Method for this System of Two First Order Differential Equations. For the system, x is the independent variable, and functions f_1 and f_2 are given by,

$w_1' = f_1(x, w_1, w_2) = w_2 \qquad ; w_2' = f_2(x, w_1, w_2) = 0.2(1-x)\left[1 + w_2^2\right]^{3/2}$

To start the solution, at $i = 0$, set x=0, w_1=0, w_2=0 , where, w_1 and w_2 represent approximations of y and y', respectively. Compute k_1 and k_2 for each of the equations at every step.

$k_{11} = hf_1(x, w_1, w_2) = hw_2 = 0.1 * 0 = 0$ *For the first equation*

$$k_{12} = hf_2(x, w_1, w_2) = h * 0.2(1-x)\left[1 + w_2^2\right]^{3/2} = 0.1 * 0.2 * (1-0)\left[1 + 0^2\right]^{3/2} = 0.02$$

For the second equation

$k_{21} = hf_1(x+h, w_1 + k_{11}, w_2 + k_{12}) = h(w_2 + k_{12}) = 0.1 * (0 + 0.02) = 0.002$

$$k_{22} = hf_2(x+h, w_1 + k_{11}, w_2 + k_{12}) = h * 0.2(1-(x+h))\left[1 + (w_2 + k_{12})^2\right]^{3/2}$$

$$= 0.1 * 0.2 * (1-(0+0.1))\left[1 + (0+0.02)^2\right]^{3/2} = 0.018011$$

Using these values, compute w_1 and w_2 at $i = 1$

$$w_1 = w_1 + \frac{k_{11} + k_{21}}{2} = 0 + \frac{0 + 0.002}{2} = 0.001$$

$$w_2 = w_2 + \frac{k_{12} + k_{22}}{2} = 0 + \frac{0.02 + 0.018011}{2} = 0.019005$$

Reset the value of x as $x+h$ and repeat the procedure for $i = 1$ and successively for all the steps. Note that the value of h is very coarse. It is accurate only to the second decimal place. To obtain good results (say, with a tolerance of 10^{-5}) the step size should be considerably lowered to 0.005. A better way is to employ other methods such as Runge-Kutta (order four).

The successive steps for the modified Euler's method are as shown below:

i	x	w_1	w_2	k_{11}	k_{12}	k_{21}	k_{22}
0	0	0	0	0	0.02	0.002	0.018011
1	0.1	0.001	0.019005	0.001901	0.01801	0.003702	0.016033
2	0.2	0.003801	0.036027	0.003603	0.016031	0.005206	0.014057
3	0.3	0.008205	0.051071	0.005107	0.014055	0.006513	0.012076
4	0.4	0.014015	0.064136	0.006414	0.012074	0.007621	0.010087
5	0.5	0.021032	0.075217	0.007522	0.010085	0.00853	0.008087
6	0.6	0.029058	0.084303	0.00843	0.008085	0.009239	0.006077
7	0.7	0.037893	0.091385	0.009138	0.006075	0.009746	0.004057
8	0.8	0.047335	0.096451	0.009645	0.004056	0.010051	0.00203
9	0.9	0.057183	0.099494	0.009949	0.00203	0.010152	-5.6E-19
10	1	0.067234	0.100509	0.010051	0	0.010051	-0.00203

If we use a step size of h=0.005, (after 200 steps) the solution will be $y_1(1)$=0.066897 and y(2)=0.100504

<u>Solution for the same example using Runge-Kutta method (order four):</u>

To start the solution, at $i = 0$, set x=0, w_1=0, w_2=0 , where, w_1 and w_2 represent approximations of y_1 and y_2 respectively. Compute k_1, k_2, k_3, and k_4 for each of the equations at every step.

$k_{11} = hf_1(x, w_1, w_2) = hw_2 = 0.1 * 0 = 0$ *For the first equation*

$$k_{12} = hf_2(x, w_1, w_2) = h * 0.2(1-x)\left[1+w_2^2\right]^{3/2} = 0.1 * 0.2 * (1-0)\left[1+0^2\right]^{3/2} = 0.02$$

For the second equation

$$k_{21} = hf_1(x+h/2, w_1+k_{11}/2, w_2+k_{12}/2) = h(w_2+k_{12}/2) = 0.1*(0+0.02/2) = 0.001$$

$$k_{22} = hf_2(x+h/2, w_1+k_{11}/2, w_2+k_{12}/2) = h*0.2(1-(x+h/2))\left[1+(w_2+k_{12}/2)^2\right]^{3/2}$$

$$= 0.1*0.2*(1-(0+0.1/2))\left[1+(0+0.02/2)^2\right]^{3/2} = 0.019003$$

$$k_{31} = hf_1(x+h/2, w_1+k_{21}/2, w_2+k_{22}/2) = h(w_2+k_{22}/2) = 0.1*(0+0.019003/2) = 0.000950$$

$$k_{32} = hf_2(x+h/2, w_1+k_{21}/2, w_2+k_{22}/2) = h*0.2(1-(x+h/2))\left[1+(w_2+k_{22}/2)^2\right]^{3/2}$$

$$= 0.1*0.2*(1-(0+0.1/2))\left[1+(0+0.019003/2)^2\right]^{3/2} = 0.019003$$

$$k_{41} = hf_1(x+h, w_1+k_{31}, w_2+k_{32}) = h(w_2+k_{32}) = 0.1*(0+0.019003) = 0.0019$$

$$k_{42} = hf_2(x+h, w_1+k_{31}, w_2+k_{32}) = h*0.2(1-(x+h))\left[1+(w_2+k_{32})^2\right]^{3/2}$$

$$= 0.1*0.2*(1-(0+0.1))\left[1+(0+0.019003)^2\right]^{3/2} = 0.01801$$

Using these values, compute w_1 and w_2 at $i = 1$

$$w_1 = w_1 + \frac{k_{11}+2k_{21}+2k_{31}+k_{41}}{6} = 0 + \frac{0+2*0.001+2*0.00095+0.0019}{6} = 0.000967$$

$$w_2 = w_2 + \frac{k_{12}+2k_{22}+2k_{32}+k_{42}}{6} = 0 + \frac{0.02+2*0.019003+2*0.019003+0.01801}{6} = 0.019003$$

Reset the value of x as $x+h$ and repeat the procedure for $i = 1$ and successively for all the steps. The successive steps are as shown below:

i	x	w1	w2	k11	k12	k21	k22	k31	k32	k41	k42
0	0	0	0	0	0.02	0.001	0.019003	0.00095	0.019003	0.0019	0.01801
1	0.1	0.000967	0.019003	0.0019	0.01801	0.002801	0.01702	0.002751	0.017019	0.003602	0.016031
2	0.2	0.003735	0.036023	0.003602	0.016031	0.004404	0.015044	0.004355	0.015043	0.005107	0.014055
3	0.3	0.008106	0.051066	0.005107	0.014055	0.005809	0.013066	0.00576	0.013065	0.006413	0.012074
4	0.4	0.013882	0.064131	0.006413	0.012074	0.007017	0.011081	0.006967	0.01108	0.007521	0.010085
5	0.5	0.020866	0.075212	0.007521	0.010085	0.008025	0.009087	0.007976	0.009086	0.00843	0.008085
6	0.6	0.028858	0.084298	0.00843	0.008085	0.008834	0.007082	0.008784	0.007081	0.009138	0.006075
7	0.7	0.037658	0.091379	0.009138	0.006075	0.009442	0.005067	0.009391	0.005066	0.009645	0.004056
8	0.8	0.047067	0.096445	0.009645	0.004056	0.009847	0.003044	0.009797	0.003043	0.009949	0.00203
9	0.9	0.05688	0.099489	0.009949	0.00203	0.01005	0.001015	0.01	0.001015	0.01005	0
10	1	0.066897	0.100504	0.01005	0	0.01005	-0.00102	0.01	-0.00102	0.009949	-0.00203

Note that the solution is accurate to the fifth decimal place even with ten steps and a relatively large step size.