SOME LAPLACE TRANSFORM EXAMPLES WITH SOLUTIONS AND COMMON TRANSFORMS

ENG 6055 MARINE CYBERNETICS

1. Examples

(1) Using the tables below show that
\[ e^{i\omega t} = \cos(\omega t) + i \sin(\omega t). \]

(2) Derive the Laplace transform of
\[ f(t) = 1 + 2 \sin(\omega t). \]

(3) Given \[ F(s) = \mathcal{L}(f(t)) \] show that for \[ g(t) = e^{-at}f(t), \]
\[ \mathcal{L}(g(t)) = G(s) = F(s + a). \]

(4) Using the above property what is the Laplace transform of
\[ f(t) = A \sin(\omega t)e^{-at}? \]

(5) Find the Laplace transform of
\[ g(t) = \frac{d}{dt} \cos(\omega t). \]

(6) Using Laplace transforms find the solution to
\[ \ddot{x} + 5 \dot{x} + 4x = 3 \] with \( x(0) = 1 \) and \( \dot{x}(0) = 1 \).

(7) Using Laplace transforms find the solution to
\[ \ddot{x} + 5 \dot{x} + 4x = u(t) \] with \( x(0) = 0, \dot{x}(0) = 0 \) and \( u(t) = 2e^{-2t}. \)

(8) What are the zeros and poles of the above transfer-function \[ G(s) = \frac{X(s)}{U(s)}. \]

1.1. Solutions.

(1) By direct computation (example given in class).

(2)
\[ F(s) = \frac{s^2 + 2\omega s + \omega^2}{s^3 + \omega^2 s} \]

(3)
\[ G(s) = \int_0^\infty g(t)e^{-st}dt = \int_0^\infty e^{-at}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s+a)t}dt = F(s + a) \]

Note: This is also called the frequency shift property.
\( F(s) = \frac{A\omega}{(s + a)^2 + \omega^2}. \)

(5) Since \( \cos(\omega 0) = 1 \) we get

\[ G(s) = \mathcal{L}(g(t)) = s\mathcal{L}(\cos(\omega t)) - \cos(\omega 0) = -\frac{\omega^2}{s^2 + \omega^2}. \]

(6) Applying the Laplace transform to the differential equation results in

\[ s^2 X(s) - sx(0) - \dot{x}(0) + 5[sX(s) - x(0)] + 4X(s) = \frac{3}{s}. \]

Using partial fraction expansion and applying the inverse Laplace transform to the result yields the following solution to the differential equation:

\[ x(t) = \frac{3}{4} + \frac{2}{3}e^{-t} + \frac{11}{12}e^{-4t} \text{ for } t > 0. \]

(7) Applying the Laplace transform to the differential equation results in

\[ s^2 X(s) + 5sX(s) + 4X(s) = \frac{2}{s + 2}. \]

After solving for \( Y(s) \) and applying partial fraction expansion we obtain

\[ X(s) = -\frac{1}{s + 2} + \frac{2}{3} + \frac{1}{3} \]

applying the inverse Laplace transform, we get

\[ x(t) = -e^{-2t} + \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t} \text{ for } t > 0. \]

(8) The transfer function of the above system is

\[ G(s) = \frac{X(s)}{U(s)} = \frac{1}{s^2 + 5s + 4} = \frac{1}{(s + 1)(s + 4)}. \]

\( G(s) \) has no zeros and poles at \( s = -1 \) and \( s = -4 \).
2. Some useful properties

2.1. Sine and Cosine as a complex function.

\[ i = \sqrt{-1} \]

\[
\sin(\omega t) = \frac{1}{2i} (e^{i\omega t} - e^{-i\omega t})
\]

\[
\cos(\omega t) = \frac{1}{2} (e^{i\omega t} + e^{-i\omega t})
\]

2.2. Laplace transforms. Laplace transforms in this table are valid for:
\( f(t) = 0 \) for \( t < 0 \) and \( T, \omega > 0 \).

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( F(s) = \mathcal{L}(f(t)) )</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{s} )</td>
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<td>( t )</td>
<td>( \frac{1}{s^2} )</td>
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<tr>
<td>( \frac{1}{2} t^2 )</td>
<td>( \frac{1}{s^3} )</td>
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<td>( e^{at} )</td>
<td>( \frac{1}{s-a} )</td>
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<td>( te^{at} )</td>
<td>( \frac{1}{(s-a)^2} )</td>
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<tr>
<td>( \frac{1}{T} e^{-\frac{t}{T}} )</td>
<td>( \frac{1}{1 + Ts} )</td>
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<td>( \frac{1}{\omega} \sin(\omega t) )</td>
<td>( \frac{1}{s^2 + \omega^2} )</td>
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<tr>
<td>( \cos(\omega t) )</td>
<td>( \frac{s}{s^2 + \omega^2} )</td>
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