Assignment 3
Due June 16, 2006 by 9:00 AM. Drop in box 89 at the general office (attention Reza Shahidi).

Question 1
10 marks

In a two-dimensional classification problem, class $C_1$ has mean \( \begin{bmatrix} a \\ 0 \end{bmatrix} \) and covariance matrix \( \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \), and class $C_2$ has mean \( \begin{bmatrix} c \\ 0 \end{bmatrix} \) and covariance matrix \( \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} \). The variables $a$, $b$, $c$, and $d$ are all greater than 0. Prove that the MICD decision boundary for this problem is a circle, except in special cases, and determine its centre and radius. Also identify the special cases and determine what the alternative decision boundary shapes are.

Question 2
30 marks

You are given the following samples that belong to a single class:

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>9</th>
<th>9</th>
<th>11</th>
<th>9</th>
<th>7</th>
<th>7</th>
<th>13</th>
<th>7</th>
<th>14</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

(a) What are the maximum likelihood estimates of the mean vector \( \mathbf{m}_1 \) and the covariance matrix \( \mathbf{S}_1 \) assuming the class p.d.f. is normal?

(b) Find the eigenvectors and eigenvalues of the class covariance matrix. Show your workings.

(c) Suppose there is a second class with mean \( \mathbf{m}_2^T = [4 \ 11] \) and the same sample covariance matrix \( \mathbf{S}_2 = \mathbf{S}_1 \). Sketch the unit standard deviation contours for both classes. Label the class means and principle axes.

(d) Sketch the MED boundaries and MICD boundaries.

(e) Assuming that both classes are equal a priori probabilities, what is the MAP classifier for the two classes? Given the point \( \mathbf{x}^T = [8 \ 6] \), to which class should it be classified?

(f) Assume the loss of classifying a pattern to $C_1$ when it really belongs to $C_2$ is 2 times the loss of classifying to $C_2$ when it truly belongs to $C_1$. Modify the MAP classifier to account for the loss. Where should the point $\mathbf{x}$ be classified now?
Question 3
10 marks

A class $C_j$ has the covariance matrix $\begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}$

What angle does the major axis of the unit standard deviation contour make with the $x_0$ axis?

Question 4 (from Duda, Hart, and Stork)
20 marks

Let $x$ have a uniform density as follows:

$$p(x | \theta) \sim U(0, \theta) = \begin{cases} \frac{1}{\theta} & 0 \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

(a) Suppose that $n$ samples $D = \{x_1, \ldots, x_n\}$ are drawn independently according to $p(x|\theta)$. Show that the maximum-likelihood estimate for $\theta$ is $\max[D]$ - that is, the value of the maximum element is $D$.

(b) Suppose that $n = 5$ points are drawn from the distribution and the maximum value of which happens to be $x_k = 0.6$. Plot the likelihood $p(D | \theta)$ in the range $0 \leq \theta \leq 1$. Explain in words why you do not need to know the value of the other four points.

Question 5 (from Duda, Hart, and Stork)
30 marks

Explore some of the properties of density estimation in the following way:

(a) Write a program to generate points according to a uniform distribution in a unit cube, $-0.5 \leq x_i \leq 0.5$ for $i = 1, 2, 3$ (i.e. 3 dimensions). Generate $10^4$ such points.

(b) Write a program to estimate the density at the origin based on your $10^4$ points as a function of the size of a cubical window function of size $h$. Plot your estimate as a function of $h$, for $0 < h \leq 1$.

(c) Write a program that generates $10^4$ points from a spherical Gaussian density (with $\Sigma = \mathbf{I}$) centered on the origin. Repeat part (b) using your Gaussian data.

(d) Discuss any qualitative differences between the functional dependencies of your estimation results for the uniform and Gaussian densities.