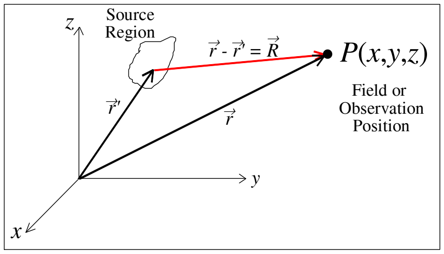
Unit 2

Coulomb's Law and Electric Field Intensity

While some properties of electricity and magnetism have been observed for many centuries, the eighteenth and nineteenth centuries really mark the beginning of the formalization of several important laws which appear to underly these properties. It must be remembered that much of what we will encounter by the way of mathematical formalization in this course is really just a means of expressing what is considered to be known fact based on experimentation. Names, such as Coulomb, Gauss, Maxwell and others with which you are already familiar from elementary physics courses, will have particularly important significance in this and/or subsequent units.

A Note on Notation:

We will have many occasions in this course to discuss *sources* and their locations and the *effects* which these sources produce at other locations which we may generally refer to as *field positions* or observation positions (these effects will include force fields, intensity fields, potential fields, etc.). The figure below emphasizes the extreme importance of having a way to distinguish position vectors associated with sources from position vectors associated with the field(s) produced by the sources.



In this course, the source region may consist of charges or currents. Generally, the positions of points within this region will labelled using "primed" coordinates as indicated by $\vec{r}\,'$ here. The positions at which the effects of these sources are measured will be indicated by unprimed coordinates, such as $\vec{r}$ here. The vector from the source to the observation or field position is then clearly $\vec{r} - \vec{r}\,'$. We will indicate the unit vector in the direction of $\vec{R}$ as $\hat{R}$. Also, $R = |\vec{r} - \vec{r}\,'|$.

2.1 Coulomb's Law

In the 1785, Charles Coulomb established that the fundamental law of *electric force* between two particles having charges of Q_1 and Q_1 possessed the following properties:

1. the *magnitude* of the force is *proportional* to the *product* of the magnitudes of the charges (i.e. $F \propto Q_1 Q_2$);
2. the *magnitude* of the force is *inversely proportional* to the the square of the distance R between the charges (i.e. $F \propto 1/R^2$);
3. the direction of the of the force is *along a line* joining the charges – this is because point charges can exert forces only *radially*;
4. the force is *attractive* if the charges are *opposite* in sign and *repulsive* if the charges have the same sign.

Putting all of this together we have

$$F \propto \frac{Q_1 Q_2}{R^2}$$

which implies that, using k as the proportionality constant and using SI units for which force is measured in **newtons** (N), charge in **coulombs** (C) and distance in **metres** (m), the magnitude of $\vec{F}$ is

$$F = k\frac{Q_1 Q_2}{R^2} \tag{2.1}$$

where

$$k =$$

with ϵ being called the *permittivity* of the region in which the charges exist. This parameter has units of **farads per metre** (F/m), the farad being the unit of capacitance. For the special case of free space we subscript the permittivity with a 0 and its value becomes

$$\epsilon_0 = 8.854 \times 10^{-12}\ \text{F/m} \approx \frac{10^{-9}}{36\pi}\ \text{F/m}\ .$$

The Vector Form of Coulomb's Law:

Consider properties (3) and (4) above in association with the following diagram in which charges Q_1 and Q_2 are located at positions $\vec{r}_1$ and $\vec{r}_2$, respectively, from some origin O:

We label the displacement vector from Q_1 to Q_2 as $\vec{R}_{12}$ and in view of equation (2.1) and the properties indicated, the force which Q_1 exerts on Q_2 is

$$\vec{F}_{12} = \tag{2.2}$$

where $\hat{R}_{12}$ is the unit vector in the direction of $\vec{R}_{12}$. Explicitly,

$$\hat{R}_{12} = \tag{2.3}$$

which means that equation (2.2) may be written as

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon} \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} \tag{2.4}$$

Of course, from Newton's third law of motion, the force which Q_2 exerts on Q_1 – i.e. $\vec{F}_{21}$ – is equal in magnitude but opposite in direction to that which Q_1 exerts on Q_2. That is,

$$\vec{F}_{21} = -\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon R_{21}^2} \hat{R}_{21} \tag{2.5}$$

with

$$\hat{R}_{21} = -\hat{R}_{12} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} \ .$$

Force on a Charge Due to Multiple Charges:

Next, suppose that a total of n charges exist in a region. We may label them Q_1, Q_2, $\ldots Q_n$ as shown:

Next, the displacement vector from the mth charge, Q_m, to Q_1 may be defined as $\vec{R}_{m1}$ with a corresponding unit vector $\hat{R}_{m1}$. Then the *total* force, $\vec{F}_T$, on Q_1 due to the other $n-1$ charges is simply the vector sum of the forces exerted by the individual charges. That is,

$$\vec{F}_T = \sum_{m=2}^{n} \frac{Q_1 Q_m}{4\pi\epsilon R_{m1}^2} \hat{R}_{m1} \tag{2.6}$$

EXAMPLE:

Four 10 μC charges are located in free space at $(-3, 0, 0)$, $(3, 0, 0)$, $(0, -3, 0)$, and $(0, 3, 0)$ in the cartesian coordinate system. Find the force on a 20 μC charge located at $(0, 0, 4)$.

2.2 Electric Field Intensity Due to Point Sources

2.2.1 Electric Field Due to a Single Point Charge

By definition, the **electric field intensity**, $\vec{E}$, at a point in space due to a **source** is the *force per unit charge* experienced by a **positive test charge**, Q_t, brought to that point. Suppose, for example that the source is a point charge Q_1. See illustration below:

In this case, from equation (2.2), on letting $Q_2 = Q_t$, the force experienced by the test charge due to source Q_1 would be

$$\vec{F}_{1t} = \frac{Q_1 Q_t}{4\pi\epsilon R_{1t}^2}\hat{R}_{1t} \ , \tag{2.7}$$

and the direction is along the line from Q_1 to Q_t. Then, from the definition, $\vec{E}$ we have

$$\vec{E} = \frac{\vec{F}_{1t}}{Q_t} = \frac{Q_1}{4\pi\epsilon R_{1t}^2}\hat{R}_{1t} \ . \tag{2.8}$$

Notice that the electric field intensity does not depend on the test charge. Therefore, it is convenient to drop the t subscripts along with the 1 subscript, since there is only one charge here (i.e. we may as well call the single charge Q) and write

$$\vec{E} = \frac{Q}{4\pi\epsilon R^2}\hat{R} \tag{2.9}$$

where $\hat{R}$ points from Q to the position where we are considering the test charge to be or, in other words, the position at which the field is being measured. If Q is at the origin of the spherical cordinate system, it also makes sense to simply use r instead of R. We note that from the definiton of $\vec{E}$ as a force per charge, the units must be

newtons/coulomb (N/C). However, you have seen in earlier courses that a **joule** (J) is a **newton·metre** which makes the newton a joule/metre. Thus,

$$\mathrm{N/C} = (\mathrm{J/m})/\mathrm{C} =$$

and since as you know (and as we will see again in this course) the (J/C) is named the **volt**, it makes good sense to use units of **V/m** for the electric field intensity.

To help with a discussion of more complicated cases than the field due to a single point charge, we may formalize equation (2.9) in terms of the diagram on page 2 of this unit. Consider that the charge Q is at position $\vec{r}'$ – i.e. at the source position – and that we wish to measure the field $\vec{E}$ at the field or observation position $\vec{r}$.

Clearly, $\vec{R} = \vec{r} - \vec{r}'$, as shown. In terms of magnitudes, $|\vec{R}| = R = |\vec{r} - \vec{r}'|$ and, as before,

$$\hat{R} = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|} \, ,$$

so that equation (2.9) may be written in general form as

$$\vec{E}(\vec{r}) = \frac{Q(\vec{r} - \vec{r}')}{4\pi\epsilon|\vec{r} - \vec{r}'|^3} \, . \tag{2.10}$$

We have written $\vec{E}$ explicitly as $\vec{E}(\vec{r})$ to emphasize that the electric field intensity depends on the position at which it is measured.

2.2.2 Electric Field Due to Multiple Point Charges

If several point charges are responsible for the electric field intensity at a partical position in space, the total field is simply the *vector sum* of the individual fields from each of the charges. Consider the following illustration while keeping an eye on equation (2.10):

Symbolizing the total field at position $\vec{r}$ as $\vec{E}(\vec{r})$, we have for the n charges

$$\vec{E}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) + \cdots + \vec{E}_n(\vec{r}) \ . \tag{2.11}$$

Now instead of writing primed vectors for the positions of the individual sources (charges), we may replace the $\vec{r}\,'$ for $Q_1, Q_2, \ldots, Q_n$ by $\vec{r}_1, \vec{r}_2, \ldots \vec{r}_n$, respectively. From equation (2.10), which gives the contribution to the field for one charge, and equation (2.11) which tells us how each charge contributes to the total field, we may immediately write

$$\vec{E}(\vec{r}) = \frac{Q_1(\vec{r}-\vec{r}_1)}{4\pi\epsilon|\vec{r}-\vec{r}_1|^3} + \frac{Q_2(\vec{r}-\vec{r}_2)}{4\pi\epsilon|\vec{r}-\vec{r}_2|^3} + \cdots + \frac{Q_n(\vec{r}-\vec{r}_n)}{4\pi\epsilon|\vec{r}-\vec{r}_n|^3} \tag{2.12}$$

or

$$\vec{E}(\vec{r}) = \sum_{m=1}^{n} \frac{Q_m(\vec{r}-\vec{r}_m)}{4\pi\epsilon|\vec{r}-\vec{r}_m|^3} \ . \tag{2.13}$$

EXAMPLE:

Four 10 μC charges are located in free space at $(-3,0,0)$, $(3,0,0)$, $(0,-3,0)$, and $(0,3,0)$ in the cartesian coordinate system. Find the electric field intensity at position $(0,0,4)$. (See problem on page 4 of this unit.)

2.3 Electric Field Intensity Due to Continuous Charge Distributions

2.3.1 Continuous Charge Distributions

When the charges are not discrete entities, but rather continuous in nature, it should come as no suprise that the effects of all the differential pieces of the charges must be summed – i.e. *integrated* – in order to find the total field effect. Before calculating the electric field intensities from such sources, however, it may be worthwhile to consider the sources themselves.

Continuous line sources:

Suppose that a line of charges has a *constant* density ρ_L (with units of C/m). Then in a distance ℓ along the line, since by definition the constant density is given by

$$\rho_L = \frac{\text{charge along } \ell}{\ell} \; ,$$

the charge Q is simply $Q = \rho_L \ell$.

Next, suppose that the charge density is a function of the position along the line. To help with the discussion, let's suppose the charge is along the z axis and the charge density is a function of the position on the axis. Since this is a *source* we use $\rho_L(z')$ as the symbol for the charge density – in general the symbol would be $\rho_L(\vec{r}\,')$. Now, by definition, an incremental length $\Delta z'$ will contain an incremental charge ΔQ. Then in keeping with our knowledge of limits, we may write that at any position z',

$$\rho_L(z') = \lim_{\Delta z' \to 0} \frac{\Delta Q}{\Delta z'} =$$

Thus, integrating gives

$$\int dQ = \int \rho_L(z')dz' \; ,$$

and over a length ℓ

$$Q = \int_\ell \rho_L(z')dz' \tag{2.14}$$

It is common practice to drop the argument from the density expression and simply use

$$Q = \int_{\ell} \rho_L dz' \ , \tag{2.15}$$

but it is important to realize that sometimes ρ_L will not be constant over the integration path.

Example: Determine the total charge from $y' = 2$ to $y' = 4$ for a charge density along the y axis given as $\rho_L = e^{-|y'|}$.

Continuous volume sources:

In equation (2.14), the charge density may change incrementally along a line. Suppose the charge density changed incrementally through a small volume, Δv located at position $\vec{r}\,'$. In this case, we speak of a volume charge density ρ_v (with units of C/m^3) defined by

$$\rho_v(\vec{r}\,') = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v'} =$$

and (2.14) is replaced by

$$Q = \int_{\text{vol}} \rho_v dv' \ , \tag{2.16}$$

where for the time being we have dropped the $\vec{r}\,'$ argument in the ρ_v function. Since this is a volume integral $\int_{\text{vol}}$ is, of course, a triple integral.

Example: Determine the total charge in the region $0 \leq x', y', z' \leq 2$ for a volume charge density given as $\rho_v = x'y'z'$.

Continuous surface sources:

Finally, suppose the charge density existed continuously over a surface. For this case, the charge density will be a *surface charge density*, ρ_S (with units of C/m^2), where again we have dropped the $\vec{r}'$ argument. This time, the integration, analogous to that in equation (2.14), will be over the surface (in general a double integral) and is given by

$$Q = \int_{\mathrm{S}} \rho_S dS' \ , \tag{2.17}$$

Example: Determine the total charge in the region $0 \leq \rho' \leq 2$, $0 \leq \phi' \leq \pi/2$ for a surface charge density given as $\rho_S = \sin\phi'$.

2.3.2 Electric Field Intensity of a Continuous Line Source

Consider a line of charges having a *uniform* charge density ρ_L along the z-axis in **free space** as shown below.:

In considering many electromagnetics problems, a good deal of work can be circumvented by first considering the the following two questions:
(1) are there any coordinates with which the field *does not* vary? or equivalently, what are the coordinates with which the field does vary?; (2) which components of the field *are not* present? or equivalently, which components of the field are present?

With respect to the figure above, we address question (1): We readily note that for a fixed ρ and z, the line charge "looks" the same no matter what the value of angle ϕ. We say that there is *azimuthal symmetry* present and $\vec{E}$ for this charge configuration does not depend on ϕ. Next, we note that if we fix ρ and ϕ and move up and down in the z direction, at any position z recedes to infinity in both positive and negative directions, and the problem does not change with this position – i.e. there is *axial symmetry.* Thus, for this case, the $\vec{E}$-field is not a function of z. However, if we fix ϕ and z and change ρ (i.e. increase or decrease the distance from the source) Coulomb's law might lead us to expect a variation in $\vec{E}$ with ρ. In summary, then , the answer to question (1) is that the field varies only with ρ.

Again, with respect to the figure, let's consider question (2): Since a differential amount of charge dQ produces only *radial* field components, there can be no $\hat{\phi}$ component. Of the remaining components, we note that at any particular position the $\hat{z}$ components will cancel due to the fields caused by charge elements equidistant above and below that position. Thus, only the $\hat{\rho}$ components survive for the $\vec{E}$ field and in view of our answer to question (1), that component should be a function of ρ only.

As we set up and complete the problem, some of the above features will become obvious from the math. From equation (2.14) and what precedes it, we see that a differential amount of charge is given by

$$dQ = \rho_L dz' \ .$$

This dQ gives rise to a differential amount of electric field intensity $d\vec{E}$. Writing equation (2.10) using these differential quantities, we have

$$d\vec{E}(\vec{r}) = \frac{dQ(\vec{r} - \vec{r}\,')}{4\pi\epsilon_0|\vec{r} - \vec{r}\,'|^3} = \tag{2.18}$$

Since the field does not vary with ϕ or z, without loss of genrality, we may choose a point on the y axis at which to calculate the form of the field due to dQ. We note then that

$$\vec{r}\,' = z'\hat{z} \text{ and } \vec{r} = y\hat{y} = \rho\hat{\rho}$$

so that

$$\vec{r} - \vec{r}\,' = \qquad\qquad\qquad .$$

Therefore, (on dropping the argument for compactness)

$$d\vec{E} =$$

We have already discussed that the $\hat{z}$ component will not be present in the field expression here (this could also be explicitly determined by doing a complete integration on the above $d\vec{E}$ expression). Given this fact, we need only determine the $\hat{\rho}$

component of the field intensity by considering

$$dE_\rho =$$

which means that

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho dz'}{4\pi\epsilon_0 \left(\rho^2 + z'^2\right)^{3/2}} \ .$$

This may not be an immediately-recognizable integral. Tables may be used to determine the answer as

or in vector form we have

$$\vec{E}(\rho) = \frac{\rho_L}{2\pi\epsilon_0 \rho} \hat{\rho} \ . \tag{2.19}$$

Other approaches to the integration may also be used. Pay close attention to pages 40 and 41 of the text. We note that equation (2.19) reveals that, unlike Coulomb's law for the point charge, the $\vec{E}$ field of an infinite line charge with uniform density IS NOT an inverse square relationship. Rather $E \propto (1/\rho)$.

It is also worthwhile to consider the effect on equation (2.19) if the line is not on the z axis (see text page 41).

2.3.3 Electric Field Intensity of a Sheet of Charge

Strip transmission lines (eg., traces on a printed circuit board) or parallel plate capacitors are two common electrical elements which basically consist of 'sheets' of charge. In these cases, surface charge density, ρ_S, will be an important feature. As a basic approach to what sometimes may become very difficult problems involving surface charge densities, in this subsection we consider the idealized case of a *infinite* sheet of charge existing in one of the coordinate planes in free space.

Let's seek to determine the field produced by a *positive* **uniform surface charge density**, ρ_S, existing in the entire $x = 0$ plane – i.e. in the yz plane:

$\vec{r} =$ Illustration:

$\vec{r}\,' =$

$\vec{R} =$

$\hat{R} =$

$dS\,' =$

It should not surprise us that the field cannot be a function of y or z because for ANY point we pick to observe the field, there is an infinite charge in both y and z directions. Furthermore, the y components will cancel and the z components will cancel (this is clear from symmetry). From these observations, we don't lose generality by choosing a point on the x-axis at which to evaluate the field. This, of course may also be revealed from the mathematics, if we seek to set up the problem as below.

Now, in equation (2.18), $dQ = \rho_S dS\,'$ so using the above quantities (and suppressing the argument in the E-field)

$$d\vec{E}(\vec{r}) = \frac{dQ(\vec{r} - \vec{r}\,')}{4\pi\epsilon_0 |\vec{r} - \vec{r}\,'|^3} =$$

and integrating gives

$$\vec{E} = \tag{2.20}$$

Equation (2.20) may be written and evaluated component by component:

$$\begin{aligned}\vec{E} &= \frac{\rho_S}{4\pi\epsilon_0}\left[\hat{x}\int_{z'=-\infty}^{\infty}\int_{y'=-\infty}^{\infty}\frac{x dy' dz'}{(x^2+y'^2+z'^2)^{3/2}} + \hat{y}\int_{z'=-\infty}^{\infty}\int_{y'=-\infty}^{\infty}\frac{-y' dy' dz'}{(x^2+y'^2+z'^2)^{3/2}} + \right.\\ &\left. + \hat{z}\int_{z'=-\infty}^{\infty}\int_{y'=-\infty}^{\infty}\frac{-z' dy' dz'}{(x^2+y'^2+z'^2)^{3/2}}\right] \qquad (2.21)\end{aligned}$$

Remember that as each integral is executed, only the variable associated with the differential is actually 'variable' – for example, as the dy' integral is completed the x and z' are considered constants and similarly as the dz' integral is completed the x and y' are considered constants. Before evaluating (2.21), we note that

Thus the dy' integral in the $\hat{y}$ component and the dz' integral in the $\hat{z}$ component integrate to 0 and both of these components therefore become 0. This leaves us with only the $\hat{x}$ component:

$$\vec{E} = \frac{\rho_S}{4\pi\epsilon_0}\left[\hat{x}\int_{z'=-\infty}^{\infty}\int_{y'=-\infty}^{\infty}\frac{x dy' dz'}{(x^2+y'^2+z'^2)^{3/2}}\right] .$$

This integral is NOT VERY NICE! We may use tables or software to determine it. Mathematica gives

$$\int_{z'=-\infty}^{\infty}\int_{y'=-\infty}^{\infty}\frac{x dy' dz'}{(x^2+y'^2+z'^2)^{3/2}} = 2\pi$$

which means that

$$\vec{E} = \frac{\rho_S}{4\pi\epsilon_0}\hat{x}(2\pi)$$

or

$$\vec{E} = \hat{x}\frac{\rho_S}{2\epsilon_0} \qquad (2.22)$$

Amazingly (perhaps), this result does not depend on the distance from the sheet of charge. Furthermore, the field is normal to the charge sheet. That is, in general,

$$\vec{E} = \hat{n}\frac{\rho_S}{2\epsilon_0} \qquad (2.23)$$

where $\hat{n}$ is a unit normal to the sheet pointing outward from the sheet. Thus, had we chosen a field point on the negative x-axis (or anywhere in the $x < 0$ halfspace), the answer would be

$$\vec{E} = -\hat{x}\frac{\rho_S}{2\epsilon_0} \tag{2.24}$$

Example: Parallel Plate Capacitor

Suppose a second sheet of charge parallel to that above having a *negative* charge density $-\rho_S$ is located at $x = a$. If we use subscript $+$ for the field from the positive ρ_S and subscript $-$ for the field from the negative ρ_S, we have:

Region $x > a$:

Region $x < 0$:

Region $0 < x < a$:

$\vec{E}_+ = \hat{x}\frac{\rho_S}{2\epsilon_0}$ and $\vec{E}_- = \hat{x}\frac{\rho_S}{2\epsilon_0}$ so that

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \hat{x}\frac{\rho_S}{2\epsilon_0} + \hat{x}\frac{\rho_S}{2\epsilon_0} = \hat{x}\frac{\rho_S}{\epsilon_0} \tag{2.25}$$

which is precisely the field between the plates of a parallel plate capacitor whose dielectric is free space (or to a good approximation air), provided the plate separation is small compared to the plate dimensions and the field point is well away from the edges (in reality there may be some so-called *fringing* at the edges so that outside the capacitor the field may be not be exactly zero as calculated above for the infinite sheets – still, in real life, the fields outside the capacitor can usually be considered to be negligible).

2.3.4 Electric Field Intensity of a Volume Charge

Consider, next, a volume charge density ρ_v in a region of space. From equation (2.16) we see that a differential amount of charge dQ is given by

$$dQ = \rho_v dv' \ .$$

Note the following illustration:

Using the above expression for dQ in the first equality of equation (2.18) gives the differential contribution to the electric field intensity as

$$d\vec{E}(\vec{r}) = \frac{dQ(\vec{r} - \vec{r}\,')}{4\pi\epsilon_0|\vec{r} - \vec{r}\,'|^3} = \frac{\rho_v dv'(\vec{r} - \vec{r}\,')}{4\pi\epsilon_0|\vec{r} - \vec{r}\,'|^3} \ .$$

Integrating both sides of this expression gives

$$\vec{E}(\vec{r}) = \int_{\text{vol}} \frac{\rho_v dv'(\vec{r} - \vec{r}\,')}{4\pi\epsilon_0|\vec{r} - \vec{r}\,'|^3} \tag{2.26}$$

This integral is usually very nasty and often cannot be completed in closed form.

2.4 Streamlines – Visualizing the Electric Field

Read Section 4.6 of the text.

The visualization of electric fields for all but the simplest charge configurations is indeed a difficult process. However, this process of visualization may be aided if we are content to look at only the *direction* of the field in specific planes, rather than in three dimensions. The result is a set of continuous lines referred to as *streamlines* (or flux lines, or field lines etc.) for which the field is everywhere tangent to the lines. As contour lines on a map also indicate steepness so the magnitude of the electric field is inversely proportional to the spacing of streamlines.

As a first example, consider the streamlines for a point charge Q as shown below. We know that the direction of the field is radial from the charge so the streamlines will also be radial:

We note that for line 1, $y = 0$, for line 2, $y = x$, for line 3, $x = 0$, for line 4, $y = -x$. That is, in all cases, $y = Cx$ where C is 0, 1, and -1 for lines 1, 2, and 4 respectively, and for line 3, $1/C = 0$.

More generally it may be noted that since the field is tangent to a streamline, from similar triangles it is easily seen that in the limit, we have

$$\frac{E_y}{E_x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

or

$$\frac{E_y}{E_x} = \frac{dy}{dx} \; . \tag{2.27}$$

Thus, if the functional forms of E_x and E_y are known, we may solve the differential equation to obtain y in terms of x – i.e. we may find the streamline equations. In general, the solution will be a family of such 'lines' (which of course are not necessarly 'lines' in the mathematical sense, since they could be any curved path).

Example: Exercise D2.7 (b) from text: Find the equation of the streamline that passes through the point $P(1, 4, -2)$ in the field

$$\vec{E} = 2e^{5x}\left[y(5x+1)\hat{x} + x\hat{y}\right] \ .$$