Unit 1

Coordinate Systems and Vectors

As you have encountered in elementary physics courses, it is a well established experimental fact that electric charges produce forces on other electric charges. In the terminology of this course, we say that electric charges give rise to *electric fields* and these fields produce forces on charges in a manner analogous to that by which masses producing gravitational fields exert forces on other masses. Furthermore, it is equally well known that steady electric currents (or direct currents), which consist of charges moving with a constant velocity, give rise to steady *magnetic fields.* Our discussion of *static* or stationary electric fields and *steady* magnetic fields, which forms the content of this course, will require a good knowledge of the properties of scalars and vectors as well as the ability to work within coordinate systems appropriate to the geometry of specific problems. In subsequent courses, it will be necessary to describe *time-varying* electric and magnetic fields and, again, an understanding of the properties of scalars and vectors and the usage of the optimal coordinate system for particular situations will be relevant. This unit is intended to review some of these basic concepts associated with the study of these ideas embodied in the so-called field of *electromagnetics.*

1.1 Scalars, Vectors, Cartesian Coordinates, and Vector and Scalar Products

Much of what follows in this section has been encountered earlier in the programme and is treated mainly as review here.

1.1.1 Scalars and Vectors – Relevant Definitons

1. **Definition of a *scalar quantity*:** a quantity possessing magnitude (size) only. A scalar quantity may be represented by a number, and usually a unit of measure. That is scalar quantities are represented by *scalars*. For example, we shall see that potential difference is such a quantity and a particular potential difference can be completely described by writing an expression such as 'x volts' where x is a pure number.

2. **Definition of a *vector quantity*:** a quantity possessing magnitude and direction. A vector quantity may be specified by a number (and a unit) and a direction. A *vector* is an arrow whose length represents the magnitude of the vector quantity and whose direction indicates the direction of the quantity. For example, we shall see that electric field intensity is a vector quantity.

   Note: The text uses boldface letters for vectors, but this is not convenient for note-taking. We will use an arrow over the quantity's symbol. For example, an electric field intensity will be represented as $\vec{E}$.

3. **Definition of a *unit vector*:** a vector whose magnitude is unity. We will use the ˆ over a particular symbol to indicate the unit vector. For example,

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

   is a unit vector in the direction of a vector, $\vec{r}$, whose magnitude is $|\vec{r}|$ or r. Also, note that the text uses boldface letters for unit vectors.

Various properties and operations associated with scalars and vectors are discussed in the following subsections.

1.1.2 Vectors and Vector Space – Discussion Based on the Cartesian Coordinate System

Consider the right-handed *rectangular* or *cartesian* coordinate system. A vector $\vec{r}$ may be specified in this system by its *components* along the respective axes as shown:

The x, y and z axes are mutually perpendicular. The position vector $\vec{r}$ is from the origin, O, to some arbitrary point $P(x, y, z)$ and is given by

$$\vec{r} = \vec{x} + \vec{y} + \vec{z}$$

where $\vec{x}$, $\vec{y}$, $\vec{z}$ are the vector components of $\vec{r}$ indicating how much of the vector lies along each of the respective coordinate axes. Thus, $\vec{r}$ may be given as an ordered triple (x, y, z). A collection of all such vectors which has the following five properties constitute a vector space: Let $\vec{r} = (x, y, z)$, $\vec{r}_1 = (x_1, y_1, z_1)$, and $\vec{r}_2 = (x_2, y_2, z_2)$.

1. Vector equality: $\vec{r}_1 = \vec{r}_2$ means $x_1 = x_2$, $y_1 = y_2$, and $z_1 = z_2$.
2. Vector addition: $\vec{r} = \vec{r}_1 + \vec{r}_2$ means $x = x_1 + x_2$, and so on.
3. Scalar multiplication: For a real number a, $a\vec{r} = (ax, ay, az)$ and vice versa.
4. Negative of a vector: $-\vec{r} = (-x, -y, -z)$ and vice versa.
5. Null vector: A vector $\vec{0}$ implies $x = 0$, $y = 0$, and $z = 0$ and vice versa.

Since the vector components are real numbers, the following properties also hold:

1. Vector addition is *commutative*: $\vec{r}_1 + \vec{r}_2 = \vec{r}_2 + \vec{r}_1$.
2. Vector addition is *associative*: $\vec{r}_1 + (\vec{r}_2 + \vec{r}_3) = (\vec{r}_1 + \vec{r}_2) + \vec{r}_3$.
3. Scalar multiplication is *distributive*: $a(\vec{r}_1 + \vec{r}_2) = a\vec{r}_1 + a\vec{r}_2$ and $(a+b)\vec{r} = a\vec{r} + b\vec{r}$.
4. Scalar multiplication is *associative*: $a(b\vec{r}) = (ab)\vec{r}$.

Furthermore, the null vector $\vec{0}$ and the negative of a given vector $\vec{r}$ are *unique*.

For various topics in engineering electromagnetics, the concepts presented above may be extended to include (1) complex vectors, (2) functions of vectors, and (3) any number of vector components.

1.1.3 Miscellaneous Vector Characteristics and Elementary Operations

Specification of $\vec{r}$ in Terms of the Standard Unit Vectors

Let $\hat{x}$, $\hat{y}$, and $\hat{z}$ be unit vectors pointing along the positive direction of the respective coordinate axes. Then, since $\vec{x} = x\hat{x}$, etc.,

$$\boxed{\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}}$$

Note, too, that the unit vector $\hat{r}$ in the direction of $\vec{r}$ is given simply as

$$\hat{r} =$$

where $|\vec{r}|$ may be determined from the *dot* or *inner* product between $\vec{r}$ and itself as shown shortly.

Dot Product

The dot product of two vectors $\vec{r}_1$ and $\vec{r}_2$ may be defined geometrically by

$$\boxed{\vec{r}_1 \cdot \vec{r}_2 = |\vec{r}_1||\vec{r}_2| \cos(\theta_2 - \theta_1)}$$

where $(\theta_2 - \theta_1)$ is the angle between $\vec{r}_1$ and $\vec{r}_2$. This is a *scalar* product – i.e. the result of the product is a scalar quantity. As claimed above, we note that

$$\vec{r} \cdot \vec{r} = |\vec{r}||\vec{r}| \cos 0^\circ = |\vec{r}|^2 \text{ or } r^2 \ .$$

Clearly, the dot product of $\vec{r}_1$ and $\vec{r}_2$ is simply the projection of $\vec{r}_1$ onto the direction of $\vec{r}_2$ multiplied by the magnitude of $\vec{r}_2$ or vice versa. This may be verified and illustrated as follows:

The dot product may be specified using the unit vector notation as follows:

$$\vec{r}_1 \cdot \vec{r}_2 = (x_1\hat{x} + y_1\hat{y} + z_1\hat{z}) \cdot (x_2\hat{x} + y_2\hat{y} + z_2\hat{z}) = x_1x_2 + y_1y_2 + z_1z_2$$

since $\hat{x} \cdot \hat{x} = 1 \cdot 1 \cos 0^\circ = 1$ etc.

while $\hat{x} \cdot \hat{y} = 1 \cdot 1 \cos 90^\circ = 0$ etc..

Again, using this idea

$$\vec{r} \cdot \vec{r} = x^2 + y^2 + z^2$$

so that

$$\boxed{|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}} \ .$$

Cross Product

Using the above notation recall that the cross product of two vectors denoted by

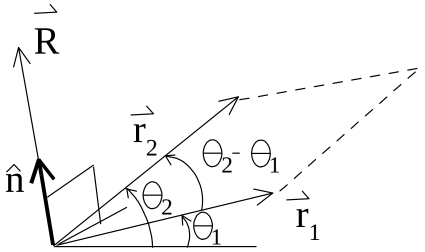
$$\boxed{\vec{R} = \vec{r}_1 \times \vec{r}_2}$$

implies

$$\boxed{|\vec{R}| = |\vec{r}_1||\vec{r}_2| \sin(\theta_2 - \theta_1)}$$

This is a *vector* product and $\vec{R}$ is specified (by definition) to be *normal* to the plane containing $\vec{r}_1$ and $\vec{r}_2$ and in the direction determine by the "right hand rule" as illustrated.

Illustration:



That is, $\vec{r}_1$, $\vec{r}_2$, and $\vec{R}$ form a right-handed system. $\hat{n}$ is a unit vector *normal* to the plane containing $\vec{r}_1$ and $\vec{r}_2$ and is in the direction of advance of a right-handed screw as $\vec{r}_1$ is rotated into $\vec{r}_2$.

Therefore,

$$\boxed{\vec{R} = \vec{r}_1 \times \vec{r}_2 = |\vec{r}_1||\vec{r}_2| \sin(\theta_2 - \theta_1)\hat{n}}$$

Clearly,

$$(\vec{r}_1 \times \vec{r}_2) = -(\vec{r}_2 \times \vec{r}_1)$$

for any pair of vectors.

Note too that $\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0$ since $\sin 0^\circ = 0$.

Also, $\hat{x} \times \hat{y} = \hat{z}$; $\hat{y} \times \hat{z} = \hat{x}$; $\hat{z} \times \hat{x} = \hat{y}$.

Determinant Form for the Cross Product:

Given: $\vec{r}_1 = x_1\hat{x} + y_1\hat{y} + z_1\hat{z}$ and $\vec{r}_2 = x_2\hat{x} + y_2\hat{y} + z_2\hat{z}$.

$\vec{r}_1 \times \vec{r}_2 =$

Therefore,

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

Of course, there is nothing special about the notation. For any two vectors, $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ and $\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$\vec{A}$ and $\vec{B}$ may be any kind of *vector* quantities (eg., force $\vec{F}$, electric field intensity, $\vec{E}$, and so on).

Scalar and Vector Fields

A *field* may be defined as some function of a vector which is specified by a connection from an arbitrary origin to a general point in space. (i.e., the field is a function of a position vector, $\vec{r}$).

A *vector field* is a vector function – i.e., it has both magnitude and direction, which will, in general, vary throughout the region.

A *scalar field* is a scalar function – i.e., it is a function of $\vec{r}$ but it has only magnitude.

It is important to realize that the value of a field may vary, in general, with time as well as position, but in this course we will consider primarily those which vary with position only.

Examples: (1) Electric Field Intensity ($\vec{E}(\vec{r})$)

The electric field intensity, $\vec{E}(\vec{r})$, of a positive point charge, q, (at $(0,0,0)$, say) is a vector field. Recall that $\vec{E}(\vec{r})$ is defined as the force per unit charge experienced by a small positive test charge brought to position $\vec{r}$.

Illustration:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}|^2}\hat{r}$$

The parameter, ϵ_0, measured in farads per metre is the *permittivity* of free space, and it has a value of $8.854 \times 10^{-12} \approx \dfrac{10^{-9}}{36\pi}$ F/m. The $\vec{E}$ field is radially away from the positive point charge and the field unit is the volt per metre (V/m). Related concepts will be discussed in much greater detail in the next unit.

(2) Electric Potential $V(\vec{r})$

We shall see later that the electric potential, V, which is defined as the work done per unit charge in bringing that charge to a position, $\vec{r}$, from some arbitrarily chosen zero reference, is strictly a *scalar* field. For example, for the above illustrated charge with the zero reference at *infinity*, it will be observed that the potential, measured in volts, is given by the scalar equation

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}|}.$$

1.2 Coordinate Systems and Coordinate Transformation

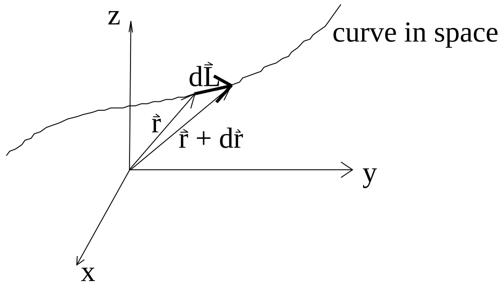
The geometry of many electromagnetics problems often dictates which coordinate system is more appropriate in facilitating a solution. For example, in examining problems associated with coaxial cables or line sources, *(circular) cylindrical coordinates* are usually useful. In other instances, certain problem symmetries may suggest the use of *spherical coordinates* and in some cases, the ordinary cartesian or rectangular system may be all that is needed. The proper choice of coordinate system is usually critical in ensuring the "quickest" solution. Of the several such systems, we'll consider only those mentioned above.

1.2.1 Cartesian (or Rectangular) Coordinates

We have already assumed knowledge of this system in the previous section. Here, we will consider specification of differential elements used in various kinds of integrations involving the (x, y, z) (i.e., Cartesian or rectangular) coordinates.

1. Specification of a Differential Length, $d\vec{L}$:

Consider a curve in space as shown:



$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

The quantity, $d\vec{L}$ is simply a differential vector of magnitude dL lying along the curve, while $\vec{r}$ and $\vec{r} + d\vec{r}$ are position vectors specifying the beginning and ending of $d\vec{L}$. Clearly, $d\vec{L} = d\vec{r}$. Therefore,

$$\boxed{d\vec{L} = dx\hat{x} + dy\hat{y} + dz\hat{z}}$$

and the magnitude is given by

$$dL = \sqrt{d\vec{L} \cdot d\vec{L}} = \sqrt{dx^2 + dy^2 + dz^2} \; .$$

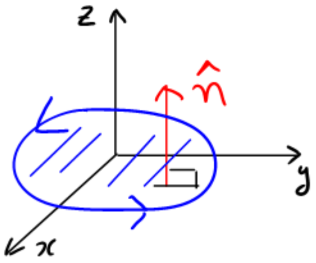
2. Specification of a Vector Differential Area, $d\vec{S}$:

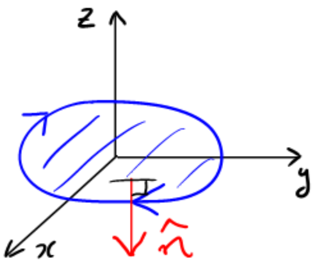
Suppose dS is a differential element of a surface which encloses a volume. Then, if $\hat{n}$ is the *outward* unit normal (positive) by definition

$$\boxed{d\vec{S} = \hat{n} dS}$$

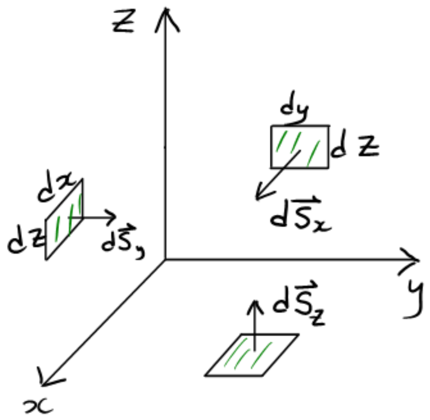
Illustration:

If the surface is not closed, the positive unit normal depends on the direction in which the perimeter is traversed. For a right-handed coordinate system this may be illustrated as





Now, consider differential areas in each of the three coordinate planes as illustrated below:



We have

$d\vec{S}_x = dydz\,\hat{x}$ is a differential surface vector along the $\hat{x}$ direction.

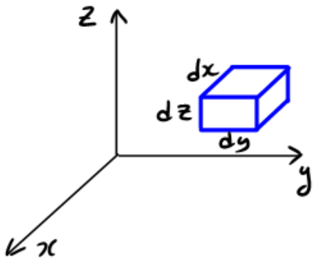
$d\vec{S}_y = dxdz\,\hat{y}$ is a differential surface vector along the $\hat{y}$ direction.

$d\vec{S}_z = dxdy\,\hat{z}$ is a differential surface vector along the $\hat{z}$ direction.

3. Specification of a Differential Volume, dv:

Consider the differential volume element, dv, in rectangular coordinates:

$$\boxed{dv = dxdydz} \; .$$

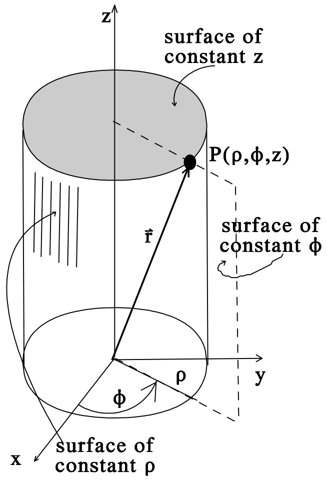


1.2.2 (Circular) Cylindrical Coordinates

We use the term "circular" *cylindrical* coordinates because, strictly speaking, there are other kinds of cylindrical coordinate systems such as "elliptic" cylindrical coordinates. Throughout this course, the term cylindical coordinates will be used in reference to the former. It is one of the two *curvilinear* coordinate systems we will encounter, the other being the *spherical* coordinate system of the next subsection.

1. Specification of the coordinate system:

Consider the following illustration:



Any point in the above illustrated space may be represented by three curvilinear coordinates which are here labelled (ρ, ϕ, z). Just as in the Cartesian system where the intersection of the planes

$$x = \text{ constant}, \quad y = \text{ constant}, \quad z = \text{ constant}$$

describes any point (x, y, z), so the intersection of the surfaces

$$\rho = \text{ constant}, \quad \phi = \text{ constant}, \quad z = \text{ constant}$$

describes any point (ρ, ϕ, z). The three coordinate surfaces are

(i.) right circular cylinders having the z-axis as the common axis.

$$\rho = \sqrt{x^2 + y^2} = \text{ constant}$$

Note that ρ is the perpendicular distance from the z-axis while r is the general distance from the origin.

(ii.) half planes through the z-axis:

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \text{ constant}$$

(ϕ is measured *counterclockwise* (ccw) from the positive x-axis as shown.)

(iii.) planes parallel to the xy-plane as in the Cartesian system:

$$z = \text{ constant}$$

The limits on ρ, ϕ and z are

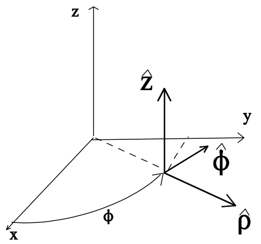
$$0 \leq \rho < \infty, \quad 0 \leq \phi \leq 2\pi, \quad -\infty < z < \infty$$

Inverting the equations under (i), (ii) and (iii) above, or going directly to the geometry of the above figure, gives the transformation from cylindrical to Cartesian cordinates as:

$$\boxed{\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}}$$

2. Unit vectors in Cylindrical Coordinates:

Consider the following illustration:



$\hat{\rho} \equiv$ unit vector normal to the cylindrical surface and pointing in the direction of increasing ρ.

$\hat{\phi} \equiv$ unit vector (tangential to the cylindrical surface), perpendicular to the half-plane $\phi =$ constant and pointing in the direction of increasing azimuth angle, ϕ.

$\hat{z} \equiv$ unit vector pointing in the direction of increasing z.

Note that

$$\hat{\rho} \cdot \hat{\rho} = \hat{\phi} \cdot \hat{\phi} = \hat{z} \cdot \hat{z} = 1$$

while

$$\hat{\rho} \times \hat{\phi} = \hat{z}; \; \hat{\phi} \times \hat{z} = \hat{\rho}; \; \hat{z} \times \hat{\rho} = \phi.$$

3. Relationships Between Cartesian and Cylindrical Unit Vectors:

It is obvious that $\hat{x}$ and $\hat{y}$, being perpendicular to $\hat{z}$, have no component in the z direction. However, both $\hat{x}$ and $\hat{y}$ may have projections along both the $\hat{\rho}$ and $\hat{\phi}$ directions (and vice versa).

Illustration: Because we are considering *unit* vectors, the following result:

$\hat{x} \cdot \hat{\rho} = \cos\phi$ $\quad$ $\hat{x} \cdot \hat{\phi} = \cos(90^\circ + \phi) = -\sin\phi$

$\hat{y} \cdot \hat{\rho} = \cos(90^\circ - \phi) = \sin\phi$ $\quad$ $\hat{y} \cdot \hat{\phi} = \cos(\phi)$

To summarize in table form:

$\cdot$	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{\rho}$	$\cos\phi$	$\sin\phi$	0
$\hat{\phi}$	$-\sin\phi$	$\cos\phi$	0
$\hat{z}$	0	0	1

On this basis,

$$\begin{aligned} \hat{x} &= (\hat{x}\cdot\hat{\rho})\hat{\rho} + (\hat{x}\cdot\hat{\phi})\hat{\phi} = \cos\phi\hat{\rho} - \sin\phi\hat{\phi} \\ \hat{y} &= \sin\phi\hat{\rho} + \cos\phi\hat{\phi} \\ \hat{z} &= \hat{z} \end{aligned}$$

with

$$\cos\phi = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \sin\phi = \frac{y}{\sqrt{x^2+y^2}}\ .$$

Similarly, it is clear from the table that

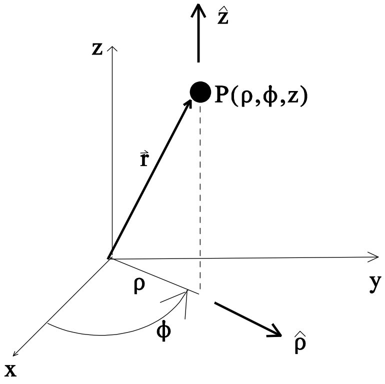
$$\begin{aligned} \hat{\rho} &= \cos\phi\hat{x} + \sin\phi\hat{y} \\ \hat{\phi} &= -\sin\phi\hat{x} + \cos\phi\hat{y} \\ \hat{z} &= \hat{z} \end{aligned}$$

4. Position Vector, $\vec{r}$, in Cylindrical Coordinates:

The above relationships between unit vectors may now be used to established the form of the position vector, $\vec{r}$, in cylindrical coordinates. We simply need to determine the projections of the Cartesian components unto the new unit vectors:

$\vec{r} =$

Of course, this result could have easily established geometrically as depicted below:



It may be noted that any general vector given by $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ may be transformed to cylindrical coordinates as $A = A_\rho\hat{\rho} + A_\phi\hat{\phi} + A_z\hat{z}$ by determining the projections of A_x and A_y unto the $\hat{\rho}$ and $\hat{\phi}$ directions. That is,

$$\vec{A} = (\vec{A}\cdot\hat{\rho})\hat{\rho} + (\vec{A}\cdot\hat{\phi})\hat{\phi} + (\vec{A}\cdot\hat{z})\hat{z}\ .$$

and

$$A_\rho = (\vec{A}\cdot\hat{\rho}) = [A_x\hat{x} + A_y\hat{y}]\cdot\hat{\rho}$$
$$A_\phi = (\vec{A}\cdot\hat{\phi}) = [A_x\hat{x} + A_y\hat{y}]\cdot\hat{\phi}$$
$$A_z = A_z.$$

5. Vector Differential Length in Cylindrical Coordinates:

Recall, in general, if u is a function of v and w, then

$$du = \frac{\partial u}{\partial v}dv + \frac{\partial u}{\partial w}dw$$

It is left as an exercise to verify, using partial derivatives and the relationships between the unit vectors, that $d\vec{r} = d\vec{L} = dx\hat{x} + dy\hat{y} + dz\hat{z}$ transforms to

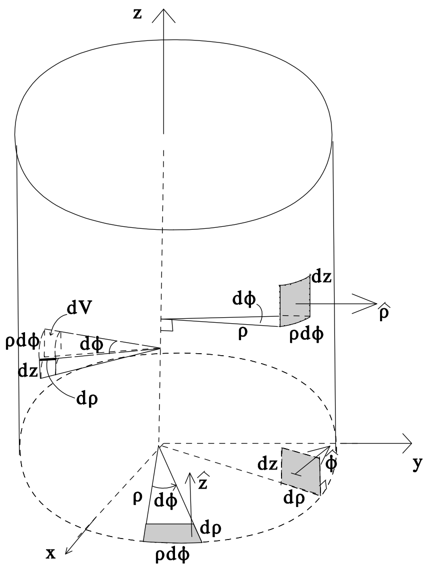
$$\boxed{d\vec{r} = d\vec{L} = d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{z}}$$

(Remember that both x and y are functions, in general, of ρ and ϕ.) Also, the above result may be quickly established from simple geometry. Finally, the magnitude of the differential length is given by

$$\boxed{dL = \sqrt{d\vec{L}\cdot d\vec{L}} = \sqrt{(d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2}}$$

6. Vector Differential Area and Differential Volume in Cylindrical Coordinates:

Consider the following diagram:



We note that

$$\boxed{\begin{array}{ll} d\vec{S}_p = \rho d\phi dz\hat{\rho} & \text{(vector differential area in } \hat{\rho} \text{ direction)} \\ d\vec{S}_\phi = d\rho dz\hat{\phi} & \text{(vector differential area in } \hat{\phi} \text{ direction)} \\ d\vec{S}_z = \rho d\rho d\phi\hat{z} & \text{(vector differential area in } \hat{z} \text{ direction)} \end{array}}$$

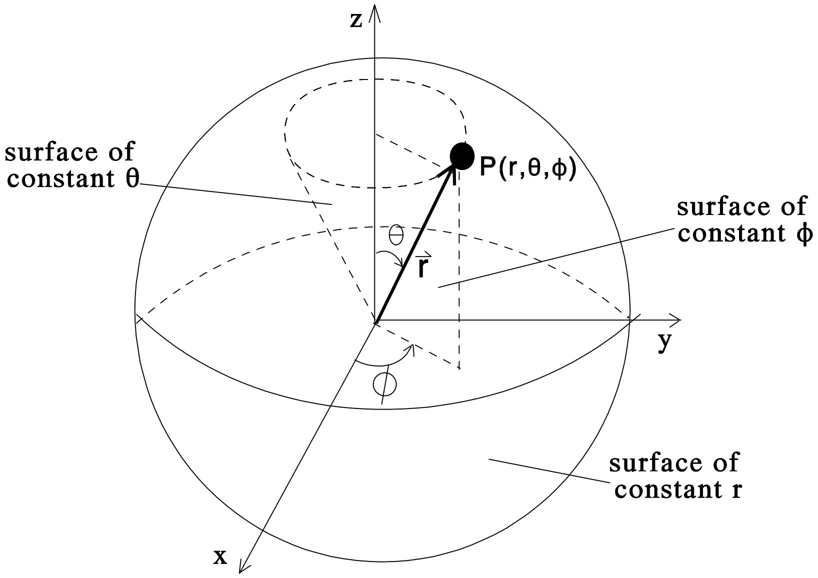
It is easily deduced from the above diagram that the differential volume element in cylindrical coordinates is given by

$$\boxed{dv = \rho d\rho d\phi dz}$$

1.2.3 Spherical Polar Coordinates

1. Specification of the coordinate system:

Consider the following illustration in which a general point, P, in space is located on a spherical surface centred at $(0,0,0)$ of the Cartesian coordinate system.



Point P may be specified by the three *spherical polar coordinates* (we'll use, simply, "spherical coordinates" in our terminology) (r,θ,ϕ). A point is uniquely specified by the intersection of the following surfaces:

(i.) Concentric spheres centres at the origin and given by

$$r=\sqrt{x^2+y^2+z^2}= \text{ constant}$$

(ii.) Right circular cones centred on the z-axis (the polar axis) with vertices at the origin and which are specified by the equation of their *polar* angle as

$$\theta=\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)= \text{ constant}$$

(iii.) Half planes through the z-axis given by

$$\phi=\tan^{-1}\left(\frac{y}{x}\right)= \text{ constant}$$

(ϕ is the azimuth angle).

The limits on r, θ and ϕ are

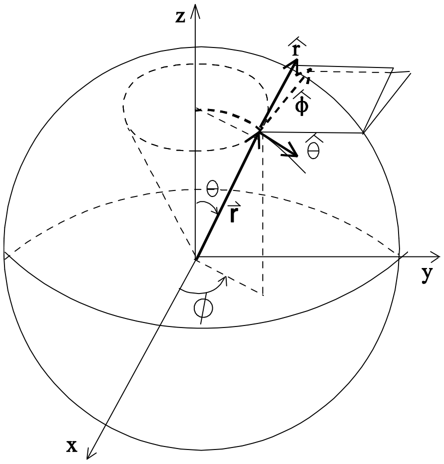
$$0 \le r < \infty, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi \ .$$

The transformation from spherical to rectangular coordinates may be readily verified from the geometry as:

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}}$$

2. Unit vectors in Spherical Coordinates:

Consider the following illustration:



$\hat{r} \equiv$ unit vector normal to the spherical surface and pointing in the direction of increasing r.

$\hat{\theta} \equiv$ unit vector tangential to the spherical surface, along a line of “longitude” as shown and pointing in the direction of increasing θ.

$\hat{\phi} \equiv$ unit vector tangential to the spherical surface, perpendicular to the half-plane $\phi =$ constant and pointing in the direction of increasing azimuth angle, ϕ.

It must be emphasized that the unit vectors $\hat{r}$ and $\hat{\theta}$ **vary in direction** as the angles θ and ϕ vary and $\hat{\phi}$ varies with ϕ.

Note that

$$\hat{r} \cdot \hat{r} = \hat{\theta} \cdot \hat{\theta} = \hat{\phi} \cdot \hat{\phi} = 1$$

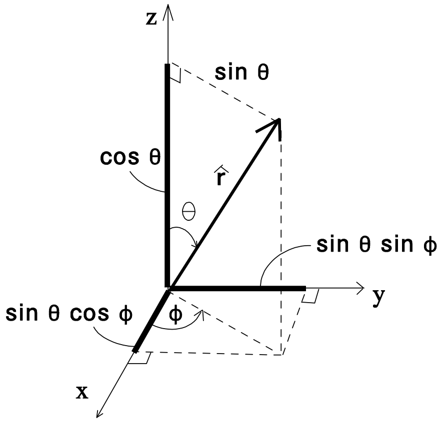
while

$$\hat{r} \times \hat{\theta} = \hat{\phi}; \ \hat{\theta} \times \hat{\phi} = \hat{r}; \ \hat{\phi} \times \hat{r} = \hat{\theta}.$$

3. Relationships Between Cartesian and Spherical Unit Vectors:

We’ll consider the ideas involved for the specific case of the projection of $\hat{r}$ onto $\hat{x}$, $\hat{y}$, and $\hat{z}$ and leave the other cases as an exercise.

Illustration: Projections of $\hat{r}$ unto the Cartesian *unit* Vectors:



A consideration of the geometry shows that the projection of $\hat{r}$ unto $\hat{x}$ is given as

$$\hat{x} \cdot \hat{r} = \sin\theta \cos\phi \ .$$

Similarly,

$$\hat{y} \cdot \hat{r} = \sin\theta \sin\phi \ \text{ and } \ \hat{z} \cdot \hat{r} = \cos\theta \ .$$

We have seen from the cylindrical coordinate discussion that the $\hat{\phi}$ projections unto the Cartesian unit vectors are given by

$$\boxed{\hat{x} \cdot \hat{\phi} = -\sin\phi \ \ ; \ \ \hat{y} \cdot \hat{\phi} = \cos\phi \ \ ; \ \ \hat{z} \cdot \hat{\phi} = 0}$$

Furthermore, it is not too difficult to set up the geometry which shows that the $\hat{\theta}$ projections are given by

$$\boxed{\begin{array}{l} \hat{x} \cdot \hat{\theta} = \cos\theta \cos\phi \\ \hat{y} \cdot \hat{\theta} = \cos\theta \sin\phi \\ \hat{z} \cdot \hat{\theta} = -\sin\theta \end{array}}$$

To summarize in table form:

$\cdot$	$\hat{r}$	$\hat{\theta}$	$\hat{\phi}$
$\hat{x}$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\hat{y}$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\hat{z}$	$\cos\theta$	$-\sin\theta$	0

On this basis,

$$
\begin{aligned}
\hat{r} &= (\hat{x}\cdot\hat{r})\hat{x} + (\hat{y}\cdot\hat{r})\hat{y} + (\hat{z}\cdot\hat{r})\hat{z} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \\
\hat{\theta} &= \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} \\
\hat{\phi} &= -\sin\theta\hat{x} + \cos\theta\hat{y}
\end{aligned}
$$

$**$It is left as an exercise to similarly determine $\hat{x}$, $\hat{y}$, and $\hat{z}$ in terms of the spherical polar coordinate unit vectors.

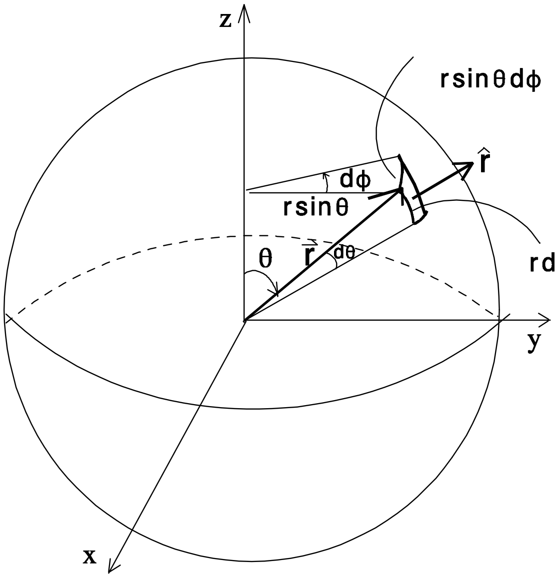
Note that by definition $\vec{r} = r\hat{r}$. This may be arrived at algebraically by using $**$ and the fact that $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ – i.e. this is a way of verifying the "correctness" of the $**$ exercise.

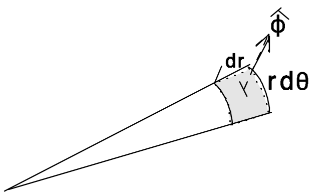
As another exercise, $**$ may be used along with the appropriate partial derivatives to show that the vector differential length is given by

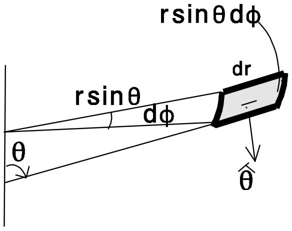
$$\boxed{d\vec{r} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}}$$

4. Vector Differential Area and Differential Volume in Spherical Coordinates:

Consider the following diagrams:







Differential Area

We note from the geometry that

$$\boxed{\begin{array}{ll} d\vec{S}_r = r^2\sin\theta d\theta d\phi\hat{r} & \text{(vector differential area in } \hat{r} \text{ direction)} \\ d\vec{S}_\theta = r\sin\theta dr d\phi\hat{\theta} & \text{(vector differential area in } \hat{\theta} \text{ direction)} \\ d\vec{S}_\phi = r dr d\theta\hat{\phi} & \text{(vector differential area in } \hat{\phi} \text{ direction)} \end{array}}$$

Differential Volume

It is easily deduced from the following diagram that the differential volume element in spherical coordinates is given by

$$\boxed{dv = r^2\sin\theta dr d\theta d\phi}$$