# Unit 3

# Electric Flux Density, Gauss's Law and Divergence

## 3.1 Electric Flux density

In (approximately) 1837, Michael Faraday, being interested in static electric fields and the effects which various insulating materials (or *dielectrics*) had on these fields, devised the following experiment:

Faraday had two concentric spheres constructed in such a way that the outer one could be dismantled into two hemispheres. With the equipment taken apart, the inner sphere was given a known *positive* charge. Then, using about 2 cm of "perfect" (ideal) dielectric material in the intervening space, the outer shell was clamped around the inner. Next, the outer shell was discharged by connecting it momentarily to ground. The outer shell was then carefully separated and the negative charge induced on each hemisphere was measured.

Faraday found that the *magnitude* of the charge induced on the outer sphere was equal to the that of the charge on the inner sphere, irrespective of the dielectric used. He concluded that there was some kind of "displacement" from the inner to the outer sphere which was independent of the medium. This is now referred to severally as *displacement*, *displacement flux*, or, as we shall use, *electric flux*. (Of course, the idea of "electric flux lines" as entities streaming away from electric charge (i.e. streamlines) is simply an invention to aid our conceptualization of the presence of an electric field). This flux, denoted by $\Psi$, is in SI units related to the charge, $Q$, producing it via a dimensionless proportionality constant of unity; i.e. the electric flux in coulombs is given by

$$\Psi = Q \quad .$$

Thus the flux is independent of the properties of the medium between the spheres and its magnitude depends only on the size of the charge producing it.

The path of the flux lines is radially away from the inner sphere as shown:

An important entity in electromagnetics is the idea of *electric flux density*, $\vec{D}$. For example, in the above illustration, at the surface of the inner sphere,

$$\vec{D} = \frac{\Psi}{\text{area of surface}}\hat{r} = \frac{Q}{4\pi a^2}\hat{r}$$

while at the outer surface

$$\vec{D} = \frac{Q}{4\pi b^2}\hat{r} \quad .$$

Clearly, $\vec{D}$ is measured in coulombs/metre$^2$ (abbreviated C/m$^2$). If we think of the inner sphere as shrinking to a point charge, $Q$, then at a distance $r$ from the charge

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r} \quad . \tag{3.1}$$

Comparing equation (3.1) with (2.10) when considering $Q$ to be at the origin so that $\vec{r}\,' = 0$ it may be seen that, for free space,

$$\vec{D} = \epsilon_0 \vec{E} \tag{3.2}$$

Equation (3.2) is one of the important so-called *constitutive relations* which are essential in solving electromagnetics problems – and it is not restricted to point charges.

If the charge is distributed within a volume, such that the charge density is $\rho_v$, then the ideas used in developing equation (2.26) lead to a volume integral

$$\vec{D} = \int_{\text{vol}} \frac{\rho_v dv'}{4\pi R^2} \hat{R} \quad . \tag{3.3}$$

Here, as usual, $R = |\vec{r} - \vec{r}'|$ is the distance from the differential volume, $dv'$, under consideration to the point of observation, and $\hat{R}$ is the unit vector $(\vec{r} - \vec{r}')/|\vec{r} - \vec{r}'|$ in that direction.

If the region of interest is NOT effectively free space, then the *permittivity*, $\epsilon_0$, must be replaced with the permittivity, $\epsilon$, of the region. However, equation (3.3) will still hold. We reserve further comment on this for Unit 5 of these notes.

**Example:**

Determine the electric flux density in a region about a uniform line charge along the $z$ axis if the charge density is 5 $\mu$C/m. Also, how much flux leaves a 2-m length of this charge distribution?

### 3.1.1 Gauss's Law

Thinking back to Faraday's experiment, it could be observed that the shape of the source charge inside the outer sphere would not be the critical factor in inducing a $-Q$ charge on the outer sphere. In fact, the inner charged body could take any shape and if it had a charge of $+Q$ in total, this would induce a $-Q$ charge on the outer sphere. The total amount of flux in the dielectric at any distance that completely enclosed the inner charged 'object' would thus be the same irrespective of the object's shape – it could be a cubical charge or even a charge on an irregularly shaped object. Of course, the distribution of the flux lines (i.e. the 'shape' of the field or equivalently the distribution of the flux density) in the dielectric would be affected, but not the total flux.

The generalization of Faraday's experiments led to the following formalization known as *Gauss's Law*:

**The electric flux passing through any closed surface is equal to the total free charge enclosed by that surface.**

In general, the closed surface may take any form we wish to visualize – which surface shape will be more convenient to consider for a particular application of Gauss's law will usually depend on the shape of the charge distribution – more soon.

Illustration: (in terms of the vector differential area, $d\vec{S}$).

Since the differential flux, $d\Psi$, *crossing* the differential area must be the product of the normal component of $\vec{D}$ and the differential surface $d\vec{S}$:

$$d\Psi = \vec{D} \cdot d\vec{S} = |\vec{D}|\,|d\vec{S}| \cos\theta$$

Therefore,

$$\Psi = \int d\Psi = \boxed{\oint_S \vec{D} \cdot d\vec{S} = Q} \text{ Gauss's Law} \tag{3.4}$$

since $\Psi = Q$, measured in coulombs, where Q is the charge enclosed by $S$. Thus, too,

$$\boxed{\oint_S \vec{D} \cdot d\vec{S} = \int_{\text{vol}} \rho_v dv' = Q} \tag{3.5}$$

Notice the circle on the surface integral. This means, in the statement of Gauss's law, the surface over which the flux is integrated surrounds the charge – the charge is **enclosed** by the surface. This closed surface, used in the context of Gauss's law, is often referred to as a *gaussian surface.*

By convention, the **outward pointing normal** is assumed in the application of Gauss's law. Using this convention, a *negative flux* simply means that flux is *entering* the surface and a *positive flux* means flux is *exiting* the surface.

Of course, if the charge is due to a linear charge density, $\rho_L$, or a surface charge density, $\rho_S$ (surface on which the charge exists is not necessarily closed), the volume integral in equation (3.5) must be replaced by a line integral or a surface integral as follows:

- For line charges, $Q = \int_{L'} \rho_L dL'$ where $\rho_L$ is the *linear charge density* in C/m.
- For surface charges, $Q = \int_{S'} \rho_S dS'$ where $\rho_s$ is the *surface charge density* in $\text{C/m}^2$.

As intimated above, facilitating appplication of Gauss's law is dependent on a suitable choice of the closed surface for integration. In fact, if the charge distribution is known, equation (3.5) can be used to obtain $\vec{D}$ in an easy manner if it is possible to choose a closed surface, $S$, which satisfies the following two properties:

(1) $\vec{D}$ is everywhere either normal or tangential to $S$ so that $\vec{D} \cdot d\vec{S} = DdS$ or 0, respectively.

AND

(2) On that portion of $S$ where $\vec{D} \cdot d\vec{S} \neq 0$, the magnitude, $|\vec{D}| = D$ is constant.

Before considering a few important examples, we note that the flux, $\Psi$, passing through a **non-closed surface** is given simply as

$$\boxed{\Psi = \int_S \vec{D} \cdot d\vec{S}} \tag{3.6}$$

<u>**Example 1:**</u> Check Gauss's law for a point charge positive $Q$ located at the origin in free space.

First, we note that the electric flux density at a distance $r$ from the charge is given by equation (3.1) as

From the radial symmetry of the flux lines it makes sense to choose a spherical gaussian surface at whose centre the charge is located. Note this surface is not some 'real material' – it is a surface imagined as surrounding the charge:

At the surface of the sphere, we have

and the vector differential surface element for the sphere is (see Unit 1):

The integral in equation (3.4) now becomes (remembering that the whole spherical surface at a particular radius is covered by allowing $\theta$ to take values from 0 to $\pi$ while $\phi$ takes values from 0 to $2\pi$)

This shows us that $Q$ coulombs of flux are crossing the surface, but this is precisely the amount of charge enclosed by the surface – i.e. we have verified Gauss's law for the point charge. Note, too, by the way, that our choice of surface, which is spherical with the charge at the centre, fits the two properties of 'nice' gaussian surfaces suggested – i.e. (1) because the field is radial, $\vec{D}$ is everywhere normal to the spherical gaussian surface and (2) the magnitude of $\vec{D}$ is constant on this surface.

**Example 2:** An infinite uniform line charge with linear density $\rho_L$ lies along the $z$-axis. Use Gauss's law to determine $\vec{D}$ and $\vec{E}$ at a distance $\rho$ from the $z$-axis.

**Observations:** (1) In Unit 2 we discussed that, from symmetry, the field will not vary with $\phi$ or $z$ and (2) only a $\hat{\rho}$ component will be present.

**Choice of Gaussian Surface:** If we choose a cylindrical surface surrounding a length $L$ of line as shown, then $\vec{D}$ is everywhere perpendicular to the sides and parallel to the ends of the cylinder (note that we need the end caps in order to close the surface for application of Gauss's law). It will shortly become clear why we don't need to consider a cylinder of infinite length.

## Example 3: The Coaxial Cable

For reasons that will become obvious from this example, the coaxial cable, consisting of an inner conductor, separated from an outer conductor by a (ideally) non-conducting dielectric is used in many electrical circuit applications. For the moment rather than thinking of currents on the inner conductor, we will consider rather that a positive charge density $\rho_S$ exists there. See illustration:

In terms of Faraday's experiment outlined at the beginning of this unit, we may think of the outer conductor as having been discharged and therefore the outer conductor will carry a net negative charge on its inner surface equal to the net positive charge on the outer surface of the inner conductor. Each flux line from the positve charge will terminate on a negative charge on the outer conductor.

We are interested in examining the $D$-field in the following three regions: (1) in the dielectric; (2) inside the inner conductor; and (3) outside both conductors altogether.

**Region 1:** $a < \rho < b$**:** The symmetry of the problem suggest that only a $D_\rho$ component of the electric flux density will exist between the conductors and so we again choose as **the gaussian surface** a right circular cylinder of length $L$ whose radius (in terms of the above diagram) is given by $a < \rho < b$.

Applying Gauss's law to this set-up we have that

$$\oint_S \vec{D} \cdot d\vec{S} = Q_{\text{enclosed}} = Q_{\text{inner}} = \int_{z'=0}^{L} \int_{\phi=0}^{2\pi} \rho_S a d\phi dz' =$$

since $\rho = a$ on the surface of the inner conductor where the charge density $\rho_S$ exists. Notice that the gaussian surface does not enclose any of the negative charge (this is extremely important for the region presently under consideration). We observe, as for Example 2, that since $\vec{D}$ is perpendicular to the inner conductor, it is parallel to the endcaps of the gaussian surface. That is, only the sides (i.e. the lateral surface) of the gaussian surface have flux passing passing through them and the integral on the left above simply becomes

$$\int_{z=0}^{L}\int_{\phi=0}^{2\pi} D_\rho\hat{\rho}\cdot(\rho d\phi dz) = 2\pi\rho D_\rho L$$

so that putting both of the above equations together we have

and

$$\boxed{\vec{D} = D_\rho\hat{\rho} = \frac{a\rho_S}{\rho}\hat{\rho}}$$

In passing, we note that if we interpret $Q/L$ as a linear charge density $\rho_L$, then from our deliberations above $\rho_L = 2\pi a\rho_S$ and the resulting flux density could be written as

$$\boxed{\vec{D} = \frac{\rho_L}{2\pi\rho}\hat{\rho}}$$

**Region 2: $\rho < a$:** The gaussian surface will have the same shape, but now it's radius must be less than $a$. This time, the enclosed charge is obviously 0.

**Region 3: $\rho > b$:** Again, the gaussian surface will have the same shape as above, but now its radius must be greater than $b$. This time, the enclosed charge is again 0 because there is a $+Q$ on the inner conductor and a $-Q$ on the outer conductor (remember Faraday's experiment). Therefore, we again have

Putting the three results in this example together, we see that the outer conductor 'shields' the inner conductor and there is no field external to the cable (we are ignoring 'end' effects). There is also no field inside the inner conductor (if, again, the conductor is ideal).

## 3.2 Gauss's Law and the Divergence Theorem

### 3.2.1 Divergenece

In the analysis of the vector and scalar fields commonly found in the study of electromagnetics, there are several operations which involve the so-called **del operator**. In cartesian coordinates this operator takes the form

$$\vec{\nabla} = \left( \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) \ .$$

We first consider the dot product,$\vec{\nabla} \cdot \vec{A}$.

<u>Definition:</u> The **divergence** of a vector function $\vec{A}$ (i.e.$\vec{A}(x, y, z)$ in cartesian coordinates) is defined as

$$\vec{\nabla} \cdot \vec{A} = \left( \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) \cdot (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \ \ .$$

Therefore,

$$\boxed{\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}} \tag{3.7}$$

Notice that the divergence of a vector is a *scalar*! The divergence will not have such a simple form in cylindrical and spherical coordinates. **We will consider the meaning shortly**, but first consider an example:

<u>Example:</u> $\vec{E} = (x^2yz)\hat{x} + (xyz^2)\hat{y} + x\hat{z}$ .

The divergence of $\vec{E}$ ( sometimes written, div $\vec{E}$), is

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial (x^2yz)}{\partial x} + \frac{\partial (xyz^2)}{\partial y} + \frac{\partial x}{\partial z} =$$

Interpretation of the Divergence:

For the sake of simplicity, consider a vector function, $\vec{A}$, that has only an $\hat{x}$ component. From (3.7),

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} \ .$$

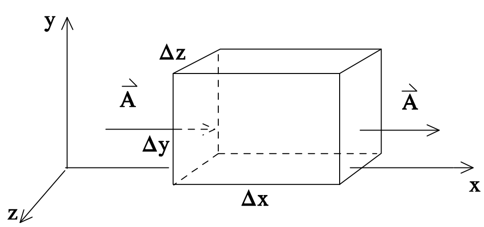
With reference to the figure below and noting that the rectangular block is of differential volume, $\left(dv = \lim_{\Delta x, \Delta y, \Delta z \to 0} \Delta x \Delta y \Delta z = \lim_{\Delta v \to 0} \Delta v\right)$, the definition of partial differentiation gives

$$\frac{\partial A_x}{\partial x} = \lim_{\Delta x \to 0} \frac{A_x\text{right face} - A_x\text{left face}}{\Delta x}$$

or

$$\frac{\partial A_x}{\partial x} = \lim_{\Delta v \to 0} \frac{A_x\text{right face}\Delta y \Delta z - A_x\text{left face}\Delta y \Delta z}{\Delta v} \ . \tag{3.8}$$

We next observe that $\lim_{\Delta y, \Delta z \to 0} \Delta y \Delta z = |d\vec{S}_x|$.




In (3.8), the first term in the numerator is the *flow out* while the second is the *flow into* the block through $dS_x$. Therefore, (3.8) may be written as

$$\frac{\partial A_x}{\partial x} = \lim_{\Delta v \to 0} \frac{\int_{S_x} \vec{A} \cdot d\vec{S}_x}{\Delta v} \tag{3.9}$$

while recalling that $d\vec{S}_x$ points *away* from the block at both the right and left faces. Making identical arguments for $\frac{\partial A_y}{\partial y}$ and $\frac{\partial A_z}{\partial z}$, (3.9) may be generalized to include all surfaces by writing

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{S}}{\Delta v} \tag{3.10}$$

where $S$ is now the whole surface of the block. Thus,

$$\vec{\nabla} \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{S}}{\Delta v} \tag{3.11}$$

and the divergence of a vector field $\vec{A}$ represents "the net outflow of the vector $\vec{A}$ per unit volume as the volume shrinks to zero". For example, $\vec{A}$ could be the electric flux density, $\vec{D}$, so that $\vec{\nabla} \cdot \vec{D}$ would be the "net flux per unit volume *leaving*" a particular region as the volume shrinks to zero. Hence the term divergence.

$\vec{\nabla} \cdot \vec{A} > 0$ if there is a net outflow (i.e. a <u>source</u> region) and

$\vec{\nabla} \cdot \vec{A} < 0$ if there is a net inflow (i.e. a <u>sink</u> region)

<u>DIVERGENCE IN CYLINDRICAL COORDINATES</u>

Starting with equation (3.11) and working backwards in cylindrical coordinates, it is not too difficult to "show" that the expression for divergence in cylindrical coordinates is:

$$\boxed{\vec{\nabla} \cdot \vec{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}} \tag{3.12}$$

<u>DIVERGENCE IN SPHERICAL COORDINATES</u>

A similar procedure in spherical coordinates leads to

$$\boxed{\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}} \tag{3.13}$$

### 3.2.2 Divergence and Maxwell's first Equation in Point Form

In equation (3.11), we now replace the general vector field $\vec{A}$ with the specific vector field, the electric flux density, $\vec{D}$. This gives

$$\vec{\nabla} \cdot \vec{D} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{D} \cdot d\vec{S}}{\Delta v} \ . \tag{3.14}$$

Since we immediately recognize from Gauss's law that

$$\oint_S \vec{D} \cdot d\vec{S} =$$

where $Q$ is the charge enclosed by $S$, the right hand side of equation (3.14) becomes

where $\rho_v$ is the familiar volume charge density. Thus, equation (3.14) may be written as

$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho_v} \quad \textbf{Maxwell's First Equation} \tag{3.15}$$

This is the first of four equations referred to as **Maxwell's equations** *as they apply to electrostatics and magnetostatics.* It is really simply Gauss's law rewritten on a per-unit-volume basis (the integral form being given by equation (3.4) or (3.5)). Here it is said to be in "point form" as it indicates that the flux per unit volume leaving (i.e. diverging from) *a vanishingly small volume* is equal to the charge per unit volume contained therein. If you simply consider the units on each side of the equation you will see this is a reasonable interpretation.

**Example: Problem 3.25, page 78 of the text. (Done in Tutorial 3)**

### 3.2.3 The Divergence Theorem

We already have the pieces of this important theorem, which will be useful throughout our study of electromagnetics. Consider the following:

First, we note that equation (3.5) may be written for *any volume* in which charge is enclosed and this volume may indeed be taken to have $S$ as the enclosing surface. Thus, we may write

$$\oint_S \vec{D} \cdot d\vec{S} = Q = \int_{\text{vol}} \rho_v dv$$

where $S$ surrounds $v$. Then we notice that equation (3.15) says

From these two equations, on replacing $\rho_v$ in the first with the divergence of $\vec{D}$ found in the second, we have

$$\boxed{\oint_S \vec{D} \cdot d\vec{S} = \int_{\text{vol}} \vec{\nabla} \cdot \vec{D} dv} \text{ The Divergence Theorem} \qquad (3.16)$$

This theorem is true for any vector field (not just $\vec{D}$). Its usefulness can be immediately seen when we consider that it is often easier to carry out a surface integral than a volume integral. What does this integral really say physically?

"It says that the flux exiting through a closed surface equals the sum total of the divergence of the flux density throughout the volume which the surface encloses." If we think of the total volume as consisting of small constitutent volumes, then as the flux diverges from one of the constituents it converges on another and so on until it reaches a constituent which contains a portion of the outer surface where at least part of the flux may *escape* – i.e. diverge – from the volume altogether. It is this escaping flux that corresponds to the *net* divergence throughout the volume. From the mathematics it is obvious that only the portion of the flux that is along a surface normal that actually contributes to the surface integral in equation (3.16). Also, see Figure 3.7, page 74 of the text. **We'll next complete Problem 3.29 of the text. (Done in Tutorial 3)**