Unit 4

Energy and Potential

To this point in our deliberations on electrostatics we have found that Coulomb's law has allowed us to generally formulate how an electric field is related to the source which produces it. These formulations are sometimes formidable when it comes to actually calculating the expressions for the field. Then, we considered how Gauss's law could be used to easily determine fields arising from 'nice' symmetrical charge distributions for which simple surfaces (gaussian surfaces) could be constructed to enclose the charge. However, once such symmetry no longer exists, Gauss law is of little help.

In this unit we consider yet a third approach to determine electric fields which involves the differentiation of a scalar function known as the *potential field* or simply *potential*. Of course, determining this new scalar field, which we will define and discuss in detail in this unit, may itself be a difficult process, but if we can find it for particular sources we shall see that the subsequent determination of the electric field is usually reasonably straightforward.

4.1 Force and Energy

Suppose that it is desired to move a charge, Q, over a displacement, $d\vec{L}$, in a region where the electric field intensity is $\vec{E}$ (which, for now, is a non-time-varying field). The force, $\vec{F}_E$, which the *field exerts* on the charge is by definition of $\vec{E}$ given as

$$\vec{F}_E = Q\vec{E} \tag{4.1}$$

By definition of *work*, the differential amount of work, dW_E, *done by the field* is given by

$$dW_E = \vec{F}_E \cdot d\vec{L}$$

and from (4.1),

$$dW_E = Q\vec{E} \cdot d\vec{L} \tag{4.2}$$

Now, if an *external* source is to move Q in the field, the the force, F_{ext}, required is equal but opposite to F_E (i.e. $F_{\text{ext}} = -\vec{F}_E$). The differential amount of *external* work, dW, must therefore be given by

and using (4.2),

$$\boxed{dW = -Q\vec{E} \cdot d\vec{L}} \tag{4.3}$$

It is clear that the *total* work done by an external force in moving a charge Q along a path, C, in a region where $\vec{E}$ exists is therefore found by adding up all the dWs along the path. That is we must integrate along C:

or

$$\boxed{W = -Q\int_C \vec{E} \cdot d\vec{L}} \tag{4.4}$$

Remember, equation (4.4) is the WORK DONE BY THE EXTERNAL SOURCE NOT BY THE FIELD ITSELF – i.e. it is the ENERGY expended by an external

source. This *definite integral* is basic to the study of field theory. If $\int_C \vec{E} \cdot d\vec{L}$ is the same for all paths C connecting two specific points (say A and B) in a field, then the field is said to be *conservative* (more on this later).

Illustration:

4.1.1 Line Integrals

The definite integral in equation (4.4) is referred to as a **line integral**. It is worth considering the meaning of this form in a bit more detail. For scalar fields, $V(\vec{r})$, say, and vector fields, $\vec{A}(\vec{r})$, say, the following forms of line integrals may be encountered:

In each case, one attempts to reduce the vector integral to scalar forms. The contour, C, is the path over which the integral is evaluated. It may be an "open" contour in which case the starting and end points are different or a "closed" contour which has the same starting and end points. For closed contours it is common to use the symbol $\oint_C$ or $\oint$.

We note that the equation (4.4) integration is of the second form indicated above where now $\vec{A}$ is replaced, in particular, with the electric field intensity $\vec{E}$. Recall that $d\vec{r} = d\vec{L}$ and that this differential length vector along C has different forms in the three coordinate systems we studied in Unit 1.

Illustration:

$\vec{E} \cdot d\vec{L}$ is "the projection of $\vec{E}$ onto $d\vec{L}$" multiplied by $|d\vec{L}|$ – i.e. $\vec{E} \cdot d\vec{L} = |\vec{E}||d\vec{L}| \cos\theta = E_L dL$ for each $d\vec{L}$ on the curve.

Thus, $\int_C \ldots = \int_B^A \vec{E} \cdot d\vec{L}$ is simply the sum of all such projections along the curve BA.

Carefully study the final example of Section 4.2 on pages 85-87 of the text. Pay particular care to the discussion on limits in the last paragraph of the Section.

Example: (a) Determine the work done in carrying a 4.0-C charge from $B(1, 0, 2)$ to $A(0.8, 0.6, 2)$ along the shorter arc of the circle given by C: $x^2 + y^2 = 1; z = 2$ if $\vec{E} = y\hat{x} + x\hat{y} + 2\hat{z}$. (b) Repeat this calculation using the straight line path from B to A.

4.2 Potential and Potential Difference

4.2.1 Potential Difference

By definition, the **potential difference** (V_{AB}) between two points in an $\vec{E}$-field is the *work per unit charge* required to move a charge **FROM** point B **TO** point A in that field. From equation (4.4),

$$\boxed{V_{AB} = \frac{W}{Q} = -\int_B^A \vec{E} \cdot d\vec{L}} \tag{4.5}$$

The unit is the joule per coulomb (J/C) or **volt** (V). The potential difference, V_{AB}, is *positive* if work is done (i.e., energy is expended by an external source) in moving the charge from B to A.

Given equation (4.5), it is clear that the potential difference between points A and B of the last example is

$$V_{AB} =$$

4.2.2 Potential (or Absolute Potential), V

The **potential**, labelled, for example, as V_A, at a point A in an electric field is the work per unit charge required to move a charge from some *arbitrarily specified zero reference* to point A. In discussing 'potential', there **must always be a reference**. Two of the most commonly used zero references are

1. the earth's surface – i.e., "ground"

and

2. infinity (especially if the earth is far removed from the region of interest).

For example, if infinity is chosen as the zero reference, using the same ideas as led to equation (1.6), we have the potential at point A as

and that at point B as

For a general reference position **ref** we may simply write

Thus,

$$V_A - V_B =$$

That is

$$\boxed{V_A - V_B = V_{AB}} \tag{4.6}$$

and hence the name "potential difference" for V_{AB}. Remember that this measurement of potential difference is **from** B **to** A.

4.3 Potential Due to Point and Continuous Charge Distributions

This section essentially contains important examples which illustrate the concepts of the previous section.

Example 1 (Point Charge): For a zero reference chosen at infinity, let's determine the potential at a distance r_A from the origin due to a point charge Q at the origin. We'll assume free space. We know for this case that the electric field intensity is

$$\vec{E}(\vec{r}) =$$

and is, of course, radial from the charge as indicated. Also, the differential path vector (for spherical coordinates) is

$$d\vec{L} = d\vec{r} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}\ .$$

Thus, since $\vec{E}$ has only an $\hat{r}$ component

$$\vec{E} \cdot d\vec{L} = \frac{Q}{4\pi\epsilon_0 r_A^2} dr$$

Therefore, from the definiton of potential,

$$V_A =$$

There is nothing special about r_A so we could write the last expression as the potential, V, using *infinity as a zero reference*, at any position r due a point charge Q at the origin is simply

$$\boxed{V(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r}} \textbf{ Potential Due to a Point Charge} \qquad (4.7)$$

Note that

- V is a function of $\vec{r}$, but it has no 'direction' asssociated with it – i.e. it is appropriate to refer to V as a *scalar field*.
- Furthermore, due to the form of $\vec{E} \cdot d\vec{L}$, we note that it is only the final value of r that affects the value of V. **The path** taken from infinity to position $\vec{r}$ **does NOT** affect the potential at $\vec{r}$. The terminology used is that the *static* electric field is **conservative**.
- Since $V(\vec{r})$ depends on how far from Q it is measured, if a charge Q_1 was at position $\vec{r}_1$ instead of being at the origin, then

$$V(\vec{r}) = \frac{Q_1}{4\pi\epsilon_0 |\vec{r} - \vec{r}_1|} \qquad (4.8)$$

- In view of the result in equation (4.7) and how we arrived at the expression for potential difference in equation (4.6), it is clear that the work done in carrying a particular charge between any two positions in a static electric force field depends on the positions but NOT ON THE PATH. A consequence of this is when the path, C, begins and ends at the same position – i.e. the path is closed – and the field is *static*, then equation (4.4) becomes

$$\oint_C \vec{E} \cdot d\vec{L} = 0 \tag{4.9}$$

  That is, the work done in carrying a charge around a closed path in the static E-field is zero. This is simply another way of formalizing the concept of a conservative field. You have previously seen a special case of this statement in electric circuits – it is, in fact, **Kirchhoff's voltage law** (without wires!). You can see this very easily by considering the units: E is in V/m and L is in metres so the integral says that the sum of the potential differences is zero for the complete loop.

- Finally, from equation (4.7) of this example and equation (4.6) of the previous section, we can simply write the potential difference between two points A and B due to a point charge at the origin is simply

$$V_{AB} = V_A - V_B = \tag{4.10}$$

Example 2 (n-Point Charges): Let's now consider the potential at position $\vec{r}$ due to n point charges scattered throughout a region of free space as shown below:

The potential, V_1, due to Q_1 at $\vec{r}_1$ is given by equation (4.8) as

$$V_1(\vec{r}) = \frac{Q_1}{4\pi\epsilon_0|\vec{r}-\vec{r}_1|} \, .$$

Of course, we have for the other charges

$$V_2(\vec{r}) = \frac{Q_2}{4\pi\epsilon_0|\vec{r}-\vec{r}_2|} \;\; ; \;\; V_3(\vec{r}) = \frac{Q_3}{4\pi\epsilon_0|\vec{r}-\vec{r}_3|} \;\; \ldots \;\; V_n(\vec{r}) = \frac{Q_n}{4\pi\epsilon_0|\vec{r}-\vec{r}_n|} \, ,$$

and the total potential at $\vec{r}$ due to all of these point charges will be

$$V(\vec{r}) =$$

We may write this result, which is linear in Q, in summation form as

$$V(\vec{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\vec{r}-\vec{r}_m|} \, . \tag{4.11}$$

Example 3 (Continuous Charge Distribution): We have seen earlier that if a charge distribution is continuous then the total charge for line, surface, and volume charge densities, is given, respectively, by equations (2.15), (2.16) and (2.17) as

$$Q = \int_\ell \rho_L dL' \, , \;\; Q = \int_S \rho_S dS' \, , \;\; Q = \int_{\text{vol}} \rho_v dv' \, .$$

Thus, if we let $\vec{r}'$ indicate the position of the source charge distributions, then we may write equation (4.11) in integral form for each kind of distribution as

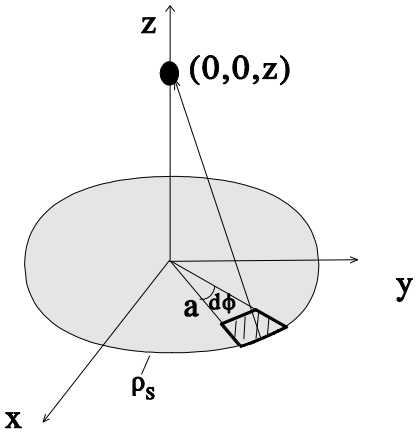
$$\boxed{V(\vec{r}) = \int_\ell \frac{\rho_L dL'}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}} \textbf{Potential Due to Line Charge} \tag{4.12}$$

$$\boxed{V(\vec{r}) = \int_S \frac{\rho_S dS'}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}} \textbf{Potential Due to Surface Charge} \tag{4.13}$$

$$\boxed{V(\vec{r}) = \int_{\text{vol}} \frac{\rho_v dv'}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}} \textbf{Potential Due to Volume Charge} \tag{4.14}$$

Carefully read page 93 of the text which gives a nice example for the potential of a ring of charge (line charge).

Example: Determine the potential at point $P(0, 0, z)$ due to the charge density ρ_S which is uniform on a circular disc of radius a centred on the origin in the x-y plane.



4.4 Potential Gradient and Equipotential Surfaces

We next seek a simple way of determining the electric field intensity, $\vec{E}$, if the potential field V is completely specified. In doing so we need to examine a mathematical entity referred to as the **gradient**. Initially, we will do this for a general *scalar* function $\varphi(x, y, z)$, and after interpreting its meaning we will make direct application to our present discussion on potential.

Definition: Consider a scalar point function $\varphi(x, y, z)$. The "del" operator, symbolized $\vec{\nabla}$, applied to this function is defined in cartesian coordinates as

$$\vec{\nabla}\varphi = \hat{x}\frac{\partial\varphi}{\partial x} + \hat{y}\frac{\partial\varphi}{\partial y} + \hat{z}\frac{\partial\varphi}{\partial z} \ ; \tag{4.15}$$

i.e.

$$\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \ . \tag{4.16}$$

The expression $\vec{\nabla}\varphi$ is referred to as the *gradient* of the scalar function φ. Thus, the application of (4.16) to a scalar function results in a vector. At this point, we issue a caution that the del operator may be treated as a vector in the gradient (and other operators, such as the divergence) ONLY in cartesian coordinates. In the other coordinate systems we have studied, this idea will require modification.

Illustration: If $f(x, y, z) = x^2yz$, $\vec{\nabla}f = (2xyz)\hat{x} + (x^2z)\hat{y} + (x^2y)\hat{z}$.

Interpretation:

Initially, let us consider the dot product

$$\vec{\nabla}\varphi \cdot d\vec{r}$$

where $d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$. Clearly,

$$\vec{\nabla}\varphi \cdot d\vec{r} = \tag{4.17}$$

by definition (i.e. the dot product in (4.17) furnishes the change in the scalar function, φ, corresponding to a change in position $d\vec{r}$).

Now the expression, $\varphi(x, y, z) = C$ where C is a constant describes some (arbitrary) *surface* in space. Consider two points, P and Q, separated by a displacement $d\vec{r}$ on such a surface. Then, moving from P to Q, the change in $\varphi(x, y, z) = C$ is given, using (4.17), by

$$d\varphi = (\vec{\nabla}\varphi) \cdot d\vec{r} = 0 \quad , \tag{4.18}$$

where the zero results because we stay on the "constant" surface. Equation (4.18) shows that $\vec{\nabla}\varphi \perp d\vec{r}$. This is true for any direction of $d\vec{r}$ from P to Q as long as the two points are restricted to the surface. Thus, $\vec{\nabla}\varphi$ is seen to be *normal* to the surface $\varphi =$ constant (A VERY IMPORTANT RESULT).

Next, as illustrated, let us permit $d\vec{r}$ to extend from one surface $\varphi = C_1$ to an adjacent surface $\varphi = C_2$ so that invoking equation (4.17)

$$d\varphi = C_2 - C_1 = \Delta C = (\vec{\nabla}\varphi) \cdot d\vec{r} = \qquad . \tag{4.19}$$

Now, for a given $d\varphi$, $|d\vec{r}|$ is a minimum when it is chosen parallel to $\vec{\nabla}\varphi$ (since then $\cos\theta = 1$); or, for a given $|d\vec{r}|$, the change in the scalar function φ is *maximized* by choosing $d\vec{r}$ parallel to $\vec{\nabla}\varphi$. This identifies $\vec{\nabla}\varphi$ as being the maximum space rate of change of the scalar φ and having a direction corresponding to that in which this maximum space rate of change occurs. We have already establish that, additionally, this direction is normal to surfaces of constant φ.

Application:

We have seen that in a conservative $\vec{E}$ field, the potential V at position $P(x, y, z)$ is unique – i.e. it is independent of the path of integration. That is V is a single-valued function $V(x, y, z)$ and the rules of calculus (such as differentiation) may be applied to it. We had that

$$V_P = -\int_{\infty}^{P} \vec{E} \cdot d\vec{L} \ .$$

Because V_P is a single-valued function, $V(x, y, z)$, we may differentiate the latter equation to give

$$\frac{dV}{dL} = -E\cos\theta$$

or, equivalently,

$$dV = -\vec{E} \cdot d\vec{L} = \qquad . \tag{4.20}$$

Comparing equations (4.19) and (4.20) immediately allows us to write that

$$\boxed{\vec{E} = -\vec{\nabla}V} \tag{4.21}$$

That is, **the magnitude of $\vec{E}$ is given by the maximum space rate of change of V, and, based on the previous discussion, the direction of $\vec{E}$ is *normal* to the *equipotential surface* (a surface where V is constant) in the direction of decreasing potential** (as indicated by the the minus sign). Equation (4.21) reads "The electric field intensity, $\vec{E}$, is the negative of the *gradient* of the potential, V". We now have a powerful way of determining the electric field once the potential field is known.

GRADIENT IN CYLINDRICAL COORDINATES

For a scalar function $\varphi(\rho, \phi, z)$ in cylindrical coordinates,

$$d\varphi = \frac{\partial \varphi}{\partial \rho} d\rho + \frac{\partial \varphi}{\partial \phi} d\phi + \frac{\partial \varphi}{\partial z} dz$$

and $d\vec{r} = d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{z}$. With a view to equation (4.17), we write

$$\begin{aligned} d\varphi &= \left(\frac{\partial \varphi}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial \varphi}{\partial \phi}\hat{\phi} + \frac{\partial \varphi}{\partial z}\hat{z} \right) \cdot d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{z} \\ &= \left(\frac{\partial \varphi}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial \varphi}{\partial \phi}\hat{\phi} + \frac{\partial \varphi}{\partial z}\hat{z} \right) \cdot d\vec{r}\,. \end{aligned}$$

Comparing this last expression with (4.32), we identify the gradient in cylindrical coordinates as

$$\vec{\nabla}\varphi = \left(\frac{\partial \varphi}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial \varphi}{\partial \phi}\hat{\phi} + \frac{\partial \varphi}{\partial z}\hat{z} \right) \tag{4.22}$$

GRADIENT IN SPHERICAL COORDINATES

A procedure completely analogous to that which led to equation (1.33) may be used when the scalar function is given in spherical coordinates as $\varphi(r, \theta, \phi)$. The result for the gradient is

$$\vec{\nabla}\varphi = \left(\frac{\partial \varphi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \varphi}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \varphi}{\partial \phi}\hat{\phi} \right) \tag{4.23}$$

4.5 The Dipole

By definition, an **electric dipole** is a pair of charges of equal magnitude but opposite sign which are separated by a distance considered to be small compared to the distance from it at which the electric and potential fields are measured. As we shall see, the concepts involved here will resurface in other contexts, both in this and subsequent courses. These contexts include ideas which range from the consideration of the behaviour of dielectrics in electric fields to the operation of antennas as transmitters and receivers of electromagnetic energy. Consider such a dipole consisting of charges $+Q$ and $-Q$ symmetrically spaced about the origin and lying on the z-axis:

Illustration:

We will seek to determine the electric field, $\vec{E}$, at P by first finding the potential, V at that point. We choose infinity as the zero reference for the potential.

STEP 1: From equation (4.8) we may write the contribution of each charge to the potential at P, in terms of the diagram above, as:

$V_1(\vec{r}) = \dfrac{Q_1}{4\pi\epsilon_0|\vec{r}-\vec{r}_1|} =$ and $V_2(\vec{r}) = \dfrac{Q_2}{4\pi\epsilon_0|\vec{r}-\vec{r}_2|} =$.

In view of equation (4.11) we therefore have $V = V_1 + V_2$ or

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) =$$

Note that when $R_1 = R_2$ (i.e. points on the x-y OR $z = 0$ plane), $V = 0$. This will be important to remember later, when we consider the concept of "images"'.

STEP 2: Next, we make use of the fact that $r \gg d$. This makes R_1 nearly parallel to R_2 and this allows us to make the approximation(see construction on diagram)

Also, in the denominator of the V equation above, to a very good approximation for the stated condition of P being distant from the dipole,

$$R_1 R_2 \approx r^2$$

where r is the distance of P from the dipole centre – i.e. from the origin. Using these latter approximations in the V equation, we get

$$V = \tag{4.24}$$

Again, notice that in the $z = 0$ plane (i.e. where $\cos\theta = 90°$) $V = 0$ as before.

STEP 3: Having obtained the potential, V, of the dipole at point P, we may now readily obtain $\vec{E}$, the electric field intensity. We note from equation (4.21), on observing that V is given in spherical coordinates, that

$$\vec{E} = -\vec{\nabla}V = \tag{4.25}$$

This gives

$$\vec{E} =$$

or

$$\vec{E} = \tag{4.26}$$

Equations (4.24) and (4.26) give us the desired fields which we will examine in a little more detail below.

Equipotential Surfaces:

In order to examine the equipotential surfaces, we turn our attention to equation (4.24). For a particular dipole, it is clear that $Qd/4\pi\epsilon_0$ is a constant, say C. Thus, that equation could be written as

$$r = \qquad .$$

If we are interested in only the shape of the equipotential surface we may set $C = 1$ and simply complete a polar plot indicated by the equation

$$r = \sqrt{\cos\theta/V}\ .$$

A plot containing a few constant V surfaces is shown on the following page. Note that when $\pi/2 < \theta < \pi$, $V < 0$. Of course r is always positive, by definition. By rotating the constant V paths about the z axis, the constant V surfaces (i.e. equipotential surfaces) result.

$\vec{E}$-Field Streamlines:

Since there are only r and θ components presentin the $\vec{E}$ field, to find the streamlines we proceed in a way exactly analogous to our discussion in Section 2.4. That is, we find

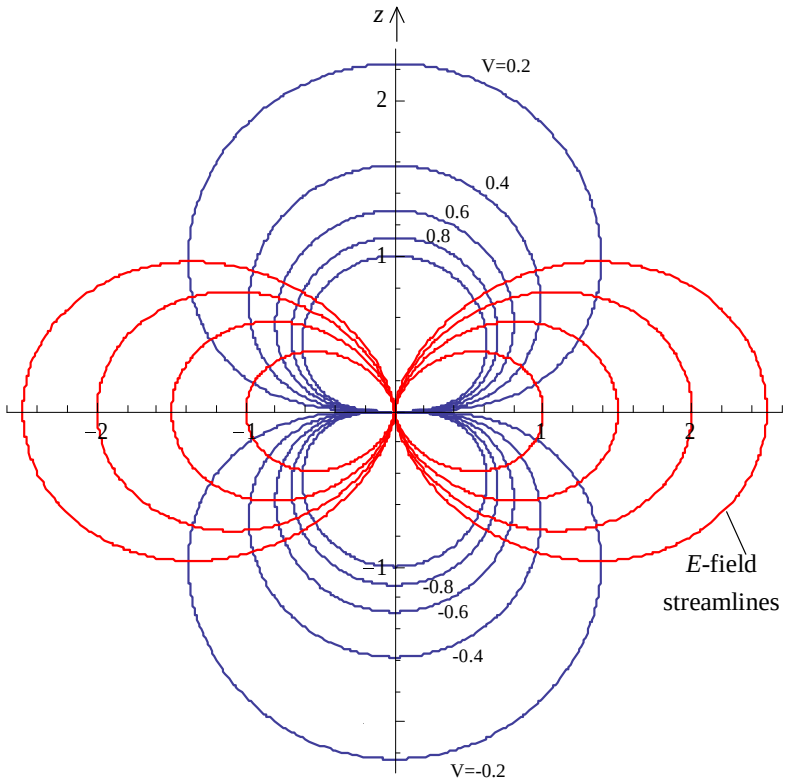
$$\frac{E_\theta}{E_r} =$$

(recall, we had $d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$ and instead of using dy/dx as we had in cartesian coordinates we now use $rd\theta/dr$). or

$$\frac{dr}{r} =$$

We seek a solution for the last equation as follows:

Equipotential Surfaces and Streamlines for Electric Dipole



Dipole Moment

With reference to the dipole shown on page 15, consider defining the **dipole moment** (symbolized as $\vec{p}$) as the product of the charge Q and the displacement vector $\vec{d}$ which points from $-Q$ to Q. That is

$$\vec{p} = \qquad . \tag{4.27}$$

The units of the dipole moment are clearly coulomb-metres ($\mathrm{C} \cdot \mathrm{m}$).

Since $\vec{d} \cdot \hat{r} = d \cos\theta$, equation (4.24) for the potential V may be written as

$$V = \qquad . \tag{4.28}$$

If the dipole is not on the z axis, but is positioned such that its centre is at $\vec{r}'$, then this last equation may be generalized to

$$V = \qquad . \tag{4.29}$$

Notice that for the dipole, $V \propto 1/r^2$ and $\vec{E} \propto 1/r^3$. Can you think of a physical reason why this would be different than the point-charge cases?

4.6 Electrostatic Field Energy Density

We now wish to consider the energy per unit volume – i.e. the **energy density** – associated with a region containing an electric field. We have already discussed that it takes work (i.e. energy is expended) to move a charge from one region of an electric field to another of different potential. This is analogous to moving a mass in a gravitational field from one potential level to another. When charges are moved to regions of higher potential in an electrostatic field, the work done is stored as energy. If the charge is released from this position, it will take on kinetic energy as it moves to regions of lower potential (just as a falling mass takes on kinetic energy as it accelerates downward in a gravitational field).

Let's consider the potential energy due to a system of charges which are brought "in" from infinity one by one to a region otherwise devoid of an electric field:

- Bringing the first charge Q_1 to its position requires no work since, at this stage of the process, there is no field present through which Q_1 is moved. Thus, we set $W_1 = 0$.

- Bringing in charge Q_2 to its position requires work due to the field produced by Q_1. From the definition of potential in equation (4.5), this work may be written as

$$W_2 = Q_2 V_{2,1}$$

  where $V_{2,1}$ is the potential at the position of Q_2 due to charge Q_1.

- Bringing in charge Q_3 to its position requires work due to the potential at that position resulting from both Q_1 and Q_2. That is

$$W_3 = Q_3 V_{3,1} + Q_3 V_{3,2} \ .$$

- extending this to the n^{th} charge, we have

$$W_n = Q_n (V_{n,1} + V_{n,2} + \ldots + V_{n,n-1}) \ .$$

The total work done, W_E, to position all charges, which is equal to the potential energy they attain, is obviously

$$W_E =$$

or

$$W_E = Q_2V_{2,1} + Q_3V_{3,1} + Q_3V_{3,2} + Q_4V_{4,1} + Q_4V_{4,2} + Q_4V_{4,3} + \ldots \tag{4.30}$$

From equation (4.8) we note, for example, that

$$Q_3V_{3,1} = Q_3\frac{Q_1}{4\pi\epsilon_0 R_{13}} = \qquad\qquad =$$

where R_{13} is the distance from Q_1 to Q_3 which is the same as R_{31}, the distance from Q_3 to Q_1. Also, $V_{1,3}$ is the potential at the position of Q_1 due to the field from Q_3. Using this last idea, equation (4.30) could be equally well written as

$$W_E = Q_1V_{1,2} + Q_1V_{1,3} + Q_2V_{2,3} + Q_1V_{1,4} + Q_2V_{2,4} + Q_3V_{3,4} + \ldots \tag{4.31}$$

Adding the left members and right members of equations (4.30) and (4.31) and factoring we have

$$\begin{aligned} 2W_E \;=\; & Q_1(V_{1,2} + V_{1,3} + V_{1,4}\ldots) \\ & +Q_2(V_{2,1} + V_{2,3} + V_{2,4} + \ldots) \\ & +Q_3(V_{3,1} + V_{3,2} + V_{3,4} + \ldots) \\ & +\ldots \\ & +Q_n(V_{n,1} + V_{n,2} + V_{n,3} + \ldots + V_{n,n-1})\ . \end{aligned}$$

Now the net potential, V_1, at the position of Q_1 due to all the other charges (except Q_1) may be written as

$$V_1 = V_{1,2} + V_{1,3} + V_{1,4} + \ldots + V_{1,n}\ .$$

Similarly defining $V_2, V_3, \ldots, V_n$, we get

$$W_E = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3 + \ldots Q_nV_n) = \tag{4.32}$$

Of course, if the charge is continuous throughout a volume (i.e. there is a volume charge density, ρ_v), the summation in equation (4.32) becomes an integral and we get

$$W_E = \tag{4.33}$$

Also, the charge densities may be line or surface charge densities, in which cases, the volume integral would become a line or surface integral, respectively, and ρ_v would be replaced by ρ_L or ρ_s, respectively. Finally, remember that both the charge density and the potential, V, may be functions of the spatial coordinates, but NOT of time (for static fields).

Next we pose the question, "Is it possible to write the energy in tems of the electric field E without direct reference to the potential V?" The answer is YES and we will now attempt to "prove" it:

Using Maxwell's first equation for electrostatic fields in point form (equation (3.15)), namely

$$\vec{\nabla} \cdot \vec{D} = \rho_v \ ,$$

equation (4.33) becomes

$$W_E = \frac{1}{2} \int_{\text{vol}} V(\vec{\nabla} \cdot \vec{D}) dv \ . \tag{4.34}$$

From the vector identity

$$\vec{\nabla} \cdot (V\vec{D}) = V(\vec{\nabla} \cdot \vec{D}) + \vec{D} \cdot (\vec{\nabla} V) \ ,$$

equation (4.34) becomes

$$W_E = \frac{1}{2} \left\{ \int_{\text{vol}} \vec{\nabla} \cdot (V\vec{D}) dv - \int_{\text{vol}} \vec{D} \cdot (\vec{\nabla} V) dv \right\} \ . \tag{4.35}$$

Using the divergence theorem of equation (3.16) in (4.35), the first volume integral becomes a *closed* surface integral, and also noting that $\vec{E} = -\vec{\nabla} V$, we may write

$$W_E = \tag{4.36}$$

In this equation, the closed surface surrounds the volume under consideration. The volume must contain all the charges giving rise to the energy. The exact size of this volume is not important and we can, in fact, let it extend to infinity, in which case

the surface S also extends to infinity. We know the following facts about the above quantities as infinity is approached:

- V approaches zero at least as fast as $1/r$ (this is the case for point charges)
- D approaches zero at least as fast as $1/r^2$
- S increases only as r^2

As a consequence the integrand and closed surface integral itself approach zero as $r \to \infty$. This leaves only

$$W_E = \frac{1}{2} \int_{\text{vol}} \vec{D} \cdot \vec{E} dv =$$

or

$$W_E = \frac{1}{2} \int_{\text{vol}} \epsilon_0 E^2 dv \ . \tag{4.37}$$

We note that since $\vec{D} \cdot \vec{E}$ has units of $(\text{C/m}^2) \cdot (\text{V/m}) \equiv (\text{C/m}^2) \cdot (\text{J/C})/\text{m} \equiv \text{J/m}^3$, the integrand represents an energy per unit volume – i.e. an **energy density** – in the field which is integrated over volume to give the total energy.

We shall consider the example of energy stored in a capacitor in a tutorial session. Also, read Section 4.8 of the text.