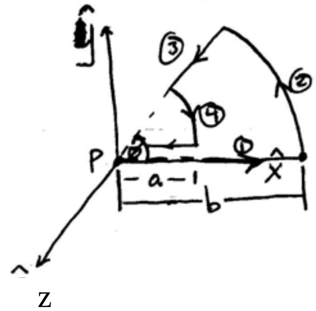
Partial Solutions to Assignment 2, Fall 2009

Question 1: A current carrying loop consisting of radial lines and circular arcs has the geometry shown. Use the Biot-Savart law to determine the magnetic field at P due to the loop current I in segments 1, 2, 3, and 4. Give your answers in terms of I, a, b, and ϕ.



Side ①: $\hat{a}_R = -\hat{\rho}$ & $d\vec{L} = d\rho\,\hat{\rho}$

Since $d\vec{L} \times \hat{a}_R = -d\rho(\hat{\rho}\times\hat{\rho}) = 0$,

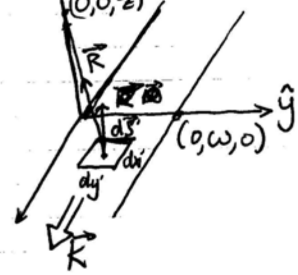
Side ① provides no contribution to $\vec{H}$ at P.

Side ②: $\hat{a}_R = -\hat{\rho}$ & $d\vec{L} = \rho\, d\phi\, \hat{\phi}$ & $\vec{R} = -b\hat{\rho}$

$$\therefore\ \vec{H}_P = \vec{H}_① + \vec{H}_② + \vec{H}_③ + \vec{H}_④ = 0 + \frac{I\phi}{4\pi b}\hat{z} + 0 + \frac{-I\phi}{4\pi a}$$

$$\boxed{\vec{H}_P = \frac{I\phi(a-b)}{4\pi ab}\ [A/m]}$$

Question 2: An infinitely long, thin conducting sheet defined over the space $-\infty \le x \le \infty$ and $0 \le y \le \omega$ is carrying a current with a uniform surface current density of $\vec{K} = \hat{x}\,5$ [A/m]. Use the Biot-Savart law to obtain an expression for the magnetic field intensity $\vec{H}$ at P(0,0,z) in cartesian coordinates



$$\vec{K} = 5\hat{x}$$

$$d\vec{s}' = dx'dy'\,\hat{z}$$

$$\vec{R} = -x'\hat{x} - y'\hat{y} + z\hat{z}$$

Note: z is not primed

$$\hat{R} = \frac{-x'\hat{x} - y'\hat{y} + z\hat{z}}{\sqrt{x'^2 + y'^2 + z^2}}$$

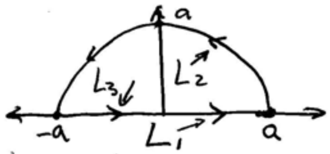
$$\vec{H} = \frac{1}{4\pi}\int_{x=-\infty}^{\infty}\int_{y=0}^{\omega} \frac{\vec{J}_s \times \hat{R}\, ds'}{R^2} \qquad ①$$

$$\text{Now } \vec{J}_s \times \hat{R} = 5\hat{x} \times \left[\frac{-x'\hat{x} - y'\hat{y} + z\hat{z}}{R}\right] = \frac{-5z\hat{y}}{R} \frac{-5y'\hat{z}}{R}$$

$$\text{This means } \vec{H} = \frac{-5}{4\pi}\int_{-\infty}^{\infty}\int_{0}^{\omega} \frac{[z\hat{y} + y'\hat{z}]}{(\sqrt{x'^2+y'^2+z^2})^3}\, dx'dy' \qquad ②$$

$$\boxed{\vec{H}_{(0,0,z)} = \frac{-5}{2\pi}\left[\hat{y}\tan^{-1}\left(\frac{\omega}{z}\right) + \hat{z}\,\frac{1}{2}\ln\left(\frac{\omega^2+z^2}{z^2}\right)\right]\ A/m}$$

Question 3: Verify Stokes' theorem over the semicircular contour shown below for the magnetic flux density:



$$\vec{B} = \hat{\rho}\,\rho\cos\phi + \hat{\phi}\sin\phi$$

Starting with: $\oint_c \vec{B}\cdot d\vec{L} = \int_{L_1} \vec{B}\cdot d\vec{L} + \int_{L_2} \vec{B}\cdot d\vec{L} + \int_{L_3} \vec{B}\cdot d\vec{L}$

First of all: $\vec{B}\cdot d\vec{L} = (\hat{\rho}\,\rho\cos\phi + \hat{\phi}\sin\phi)\cdot(\hat{\rho}\,d\rho + \hat{\phi}\,\rho\,d\phi + \hat{z}\,dz)$

$$= \rho\cos\phi\,d\rho + \rho\sin\phi\,d\phi$$

$$L_1:\ \int_{L_1} \vec{B}\cdot d\vec{L} = \int_0^a \left(\rho\cos\phi\,d\rho\Big|_{\substack{\phi=0\\ z=0}} + \int_{\phi=0}^{0} \rho\sin\phi\,d\phi\Big|_{\substack{\phi=0\\ z=0}}\right)$$

$$= \left[\tfrac{1}{2}\rho^2\right]_0^a = \tfrac{1}{2}a^2 \,/\!/$$

Do this for the remaining pieces of the path and add up the three pieces to get

$$\oint \vec{B}\cdot d\vec{L} = \tfrac{1}{2}a^2 + 2a + \tfrac{1}{2}a^2 = \boxed{a^2 + 2a}$$

Then evaluate

$$\iint_S (\vec{\nabla}\times\vec{B})\cdot d\vec{S}$$

where the surface vector is $(\hat{z}\,\rho\,d\rho\,d\phi)$
to get the same answer.

Question 4: A long cylindrical wire of radius a is centred on the z-axis. The conductor carries a current density $\vec{J} = \hat{z}\,(J_0/\rho)$ where J_0 is a constant (in units of A/m^2) and ρ is the usual cylindrical coordinate. Use Ampere's law to determine $\vec{H}$ inside and outside the conductor.

Inside the conductor:

$$I_1 = \iint_S \vec{J}\cdot d\vec{S} = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{\rho_1} \left(\frac{\hat{z}\,J_0}{\rho}\right)\cdot(\hat{z}\,\rho\,d\rho\,d\phi)$$

$$= 2\pi\int_{\rho=0}^{\rho_1} J_0\,d\rho$$

$$= 2\pi\rho_1 J_0 \ [A]$$

Since $\vec{H}_1 = \dfrac{I_1}{2\pi\rho_1}\hat{\phi} \longrightarrow \boxed{\vec{H}_1 = J_0\,\hat{\phi}}$ [A/m]

Choose a new integration path for "outside" the conductor and repeat to get

$$\boxed{\vec{H}_2 = \hat{\phi}\,J_0\left(\frac{a}{\rho}\right)\ [A/m]}$$

Question 5: A uniform current density given by:

$$\vec{J} = \hat{z} J_o \ [A/m]$$

gives rise to a vector potential

$$\vec{A} = -\hat{z}\frac{\mu_o J_o}{4}(x^2+y^2) \ [Wb/m]$$

a) Apply Poisson's equation to confirm the above claim.
b) Use $\vec{A}$ to find $\vec{H}$.
c) Use the expression for $\vec{J}$ and Ampere's law to find $\vec{H}$ and compare with the result in (b)

a) $$\nabla^2\vec{A} = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z = \hat{z}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left[-\mu_o\frac{J_o}{4}(x^2+y^2)\right]$$

$$= -\hat{z}\mu_o\frac{J_o}{4}(2+2) = \boxed{-\hat{z}\mu_o J_o}$$

Poisson's equation states that $\nabla^2 A = -\mu_o\vec{J}$ which can be seen above.

b) $$\vec{H} = \frac{1}{\mu_o}(\vec{\nabla}\times\vec{A}) = \frac{1}{\mu_o}\left[\hat{x}\left(\frac{\partial A_z}{\partial y}-\cancel{\frac{\partial A_y}{\partial z}}^{\,0}\right)+\hat{y}\left(\cancel{\frac{\partial A_x}{\partial z}}^{\,0}-\frac{\partial A_z}{\partial x}\right)+\hat{z}\left(\cancel{\frac{\partial A_y}{\partial x}}^{\,0}-\cancel{\frac{\partial A_x}{\partial y}}^{\,0}\right)\right]$$

$$= \frac{1}{\mu_o}\left(\hat{x}\frac{\partial A_z}{\partial y} - \hat{y}\frac{\partial A_z}{\partial x}\right)$$

$$= \frac{1}{\mu_o}\left(\hat{x}\frac{\partial}{\partial y}\left(-\frac{\mu_o J_o}{4}(x^2+y^2)\right)-\hat{y}\frac{\partial}{\partial x}\left(-\frac{\mu_o J_o}{4}(x^2+y^2)\right)\right)$$

$$\boxed{\vec{H} = -\hat{x}\frac{J_o y}{2} + \hat{y}\frac{J_o x}{2} \ [A/m]}$$

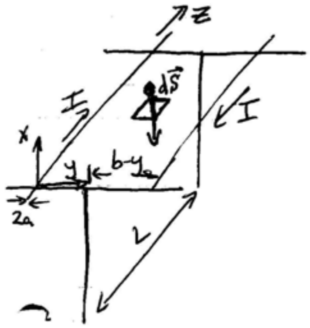
c) Ampere's Law: $\oint_c \vec{H}\cdot d\vec{L} = I = \iint_s \vec{J}\cdot d\vec{S}$

Use this to develop an answer in **cylindrical coordinates** and then use the dot product tables to convert to Cartesian coordinates to get

$$\boxed{\vec{H} = -\hat{x}\frac{yJ_o}{2} + \hat{y}\frac{xJ_o}{2} \ [A/m]}$$

which is what was derived in (b).

Question 6: A particular transmission line carries 1000A in opposite directions in each of its long parallel conductors strung on poles 100m apart. If the radius of each conductor is 2cm and the separation between their axes is 1m, determine the total flux passing through the rectangular region bounded by the conductors and two consecutive poles. See diagram below.



To find Φ passing through the rectangular region:

We know for a "long" conductor:

$$\vec{H} = \frac{I}{2\pi\rho}\hat{\phi} \rightarrow \begin{cases} \vec{H}_1 = \dfrac{-I}{2\pi y}\hat{x} \\ \vec{H}_2 = \dfrac{-I}{2\pi(b-y)}\hat{x} \end{cases} \text{ where } b = 1m$$

$$\therefore \vec{H} = \vec{H}_1 + \vec{H}_2 = \frac{-I}{2\pi}\left[\frac{1}{y} + \frac{1}{1-y}\right]\hat{x}$$

$$\vec{B} = \mu_0\vec{H} = \frac{-\mu_0 I}{2\pi}\left[\frac{1}{y} + \frac{1}{1-y}\right]\hat{x}\ ;\ d\vec{s} = -dydz\,\hat{x}$$

$$\Phi = \int_{z=0}^{l}\int_{y=a}^{1-a}\vec{B}\cdot d\vec{s}$$

$$= \int_0^{100}\int_{0.02}^{0.98}\frac{\mu_0 I}{2\pi}\left[\frac{1}{y} + \frac{1}{1-y}\right]dy\,dz$$

$$= \frac{\mu_0 I(100)}{2\pi}\left[\ln\left(\frac{y}{1-y}\right)\right]_{0.02}^{0.98}$$

$$= \frac{(\cancel{4\pi}^{2}\times 10^{-7}\,F/m)(1000A)(100m)}{\cancel{2\pi}}\left\{\ln\left[\frac{0.98}{0.02}\right] - \ln\left[\frac{0.02}{0.98}\right]\right\}$$

$$\boxed{\Phi = 1.5567\times 10^{-1}\ Wb}$$