Partial Solutions of ENGI 6813 Assignment 3, Fall 09

1.

8.20. A solid conductor of circular cross-section with a radius of 5 mm has a conductivity that varies with radius. The conductor is 20 m long and there is a potential difference of 0.1 V dc between its two ends. Within the conductor, $\mathbf{H} = 10^5\rho^2\mathbf{a}_\phi$ A/m.

a) Find σ as a function of ρ: Start by finding $\mathbf{J}$ from $\mathbf{H}$ by taking the curl. With $\mathbf{H}$ ϕ-directed, and varying with radius only, the curl becomes:

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}(10^5\rho^3)\,\mathbf{a}_z = 3 \times 10^5 \rho\,\mathbf{a}_z \text{ A/m}^2$$

Then use Ohm's law in point form to get the conductivity.

b) Find I using $\boldsymbol{J}$ and then determine R from the usual form of Ohm's law.

2.

8.26. Let $\mathbf{G} = 15r\mathbf{a}_\phi$.

a) Determine $\oint \mathbf{G} \cdot d\mathbf{L}$ for the circular path $r = 5$, $\theta = 25°$, $0 \le \phi \le 2\pi$:

$$\oint \mathbf{G} \cdot d\mathbf{L} = \int_0^{2\pi} 15(5)\mathbf{a}_\phi \cdot \mathbf{a}_\phi(5)\sin(25°)\,d\phi = 2\pi(375)\sin(25°) = \underline{995.8}$$

b) Get curl of **G** and use vector differential surface $\mathbf{dS_r}$ to do the appropriate right hand side integral of Stokes' law. The value of this integral should again be 995.8.

3. Solution not given.

4.

8.34. A filamentary conductor on the z axis carries a current of 16A in the $\mathbf{a}_z$ direction, a conducting shell at $\rho = 6$ carries a total current of 12A in the $-\mathbf{a}_z$ direction, and another shell at $\rho = 10$ carries a total current of 4A in the $-\mathbf{a}_z$ direction.

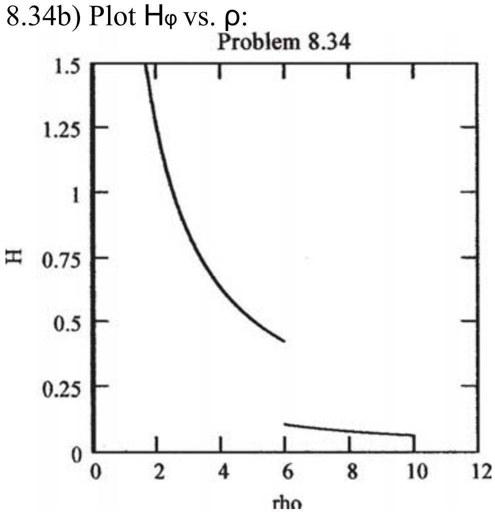
a) Find $\mathbf{H}$ for $0 < \rho < 12$: Ampere's circuital law states that $\oint \mathbf{H} \cdot d\mathbf{L} = I_{encl}$, where the line integral and current direction are related in the usual way through the right hand rule. Therefore, if I is in the positive z direction, $\mathbf{H}$ is in the $\mathbf{a}_\phi$ direction. We proceed as follows:

$$0 < \rho < 6: \quad 2\pi\rho H_\phi = 16 \quad \Rightarrow \quad \mathbf{H} = \underline{16/(2\pi\rho)\mathbf{a}_\phi}$$

$$6 < \rho < 10: \quad 2\pi\rho H_\phi = 16 - 12 \quad \Rightarrow \quad \mathbf{H} = \underline{4/(2\pi\rho)\mathbf{a}_\phi}$$

$$\rho > 10: \quad 2\pi\rho H_\phi = 16 - 12 - 4 = 0 \quad \Rightarrow \quad \mathbf{H} = \underline{0}$$

8.34b) Plot H_ϕ vs. ρ:



c) Find the total flux Φ crossing the surface $1 < \rho < 7$, $0 < z < 1$: Integrate using a *dρdz* surface integral to get <u>5.9 μWb.</u>

5.

9.1. A point charge, $Q = -0.3\mu C$ and $m = 3 \times 10^{-16}$ kg, is moving through the field $\mathbf{E} = 30\,\mathbf{a}_z$ V/m. Use Eq. (1) and Newton's laws to develop the appropriate differential equations and solve them, subject to the initial conditions at $t = 0$: $\mathbf{v} = 3 \times 10^5\,\mathbf{a}_x$ m/s at the origin. At $t = 3\,\mu s$, find:

a) the position $P(x, y, z)$ of the charge: The force on the charge is given by $\mathbf{F} = q\mathbf{E}$, and Newton's second law becomes:

$$\mathbf{F} = m\mathbf{a} = m\frac{d^2\mathbf{z}}{dt^2} = q\mathbf{E} = (-0.3 \times 10^{-6})(30\,\mathbf{a}_z)$$

describing motion of the charge in the z direction. The initial velocity in x is constant, and so no force is applied in that direction. We integrate once:

$$\frac{dz}{dt} = v_z = \frac{qE}{m}t + C_1$$

The initial velocity along z, $v_z(0)$ is zero, and so $C_1 = 0$. Integrating a second time yields the z coordinate:

$$z = \frac{qE}{2m}t^2 + C_2$$

The charge lies at the origin at $t = 0$, and so $C_2 = 0$. Introducing the given values, we find

$$z = \frac{(-0.3 \times 10^{-6})(30)}{2 \times 3 \times 10^{-16}}t^2 = -1.5 \times 10^{10}t^2 \text{ m}$$

At $t = 3\,\mu s$, $z = -(1.5 \times 10^{10})(3 \times 10^{-6})^2 = -.135$ cm. Now, considering the initial constant velocity in x, the charge in $3\,\mu s$ attains an x coordinate of $x = vt = (3\times10^5)(3\times10^{-6}) = .90$ m. In summary, at $t = 3\,\mu s$ we have $P(x, y, z) = \underline{(.90, 0, -.135)}$.

b) the velocity, $\mathbf{v}$: After the first integration in part a, we find

$$v_z = \frac{qE}{m}t = -(3 \times 10^{10})(3 \times 10^{-6}) = -9 \times 10^4 \text{ m/s}$$

Including the intial x-directed velocity, we finally obtain $\mathbf{v} = \underline{3 \times 10^5\,\mathbf{a}_x - 9 \times 10^4\mathbf{a}_z \text{ m/s}}$.

c) the kinetic energy of the charge: Have

$$\text{K.E.} = \frac{1}{2}m|v|^2 = \frac{1}{2}(3 \times 10^{-16})(1.13 \times 10^5)^2 = \underline{1.5 \times 10^{-5}\text{ J}}$$

6.

9.3. A point charge for which $Q = 2 \times 10^{-16}$ C and $m = 5 \times 10^{-26}$ kg is moving in the combined fields $\mathbf{E} = 100\mathbf{a}_x - 200\mathbf{a}_y + 300\mathbf{a}_z$ V/m and $\mathbf{B} = -3\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z$ mT. If the charge velocity at $t = 0$ is $\mathbf{v}(0) = (2\mathbf{a}_x - 3\mathbf{a}_y - 4\mathbf{a}_z) \times 10^5$ m/s:

a) give the unit vector showing the direction in which the charge is accelerating at $t = 0$: Use the force vector **F** (after finding it using the electric and magnetic fields); since this is the direction of the acceleration vector, it is simple to find a unit vector in this direction.

7.

9.5. A rectangular loop of wire in free space joins points $A(1, 0, 1)$ to $B(3, 0, 1)$ to $C(3, 0, 4)$ to $D(1, 0, 4)$ to A. The wire carries a current of 6 mA, flowing in the $\mathbf{a}_z$ direction from B to C. A filamentary current of 15 A flows along the entire z axis in the $\mathbf{a}_z$ direction.

a) Find **F** on side BC:

$$\mathbf{F}_{BC} = \int_B^C I_{\text{loop}}d\mathbf{L} \times \mathbf{B}_{\text{from wire at BC}}$$

Thus

$$\mathbf{F}_{BC} = \int_1^4 (6 \times 10^{-3})\,dz\,\mathbf{a}_z \times \frac{15\mu_0}{2\pi(3)}\,\mathbf{a}_y = -1.8 \times 10^{-8}\mathbf{a}_x \text{ N} = \underline{-18\mathbf{a}_x \text{ nN}}$$

c) Find $\mathbf{F}_{\text{total}}$ on the loop: This will be the vector sum of the forces on the four sides. Note that by symmetry, the forces on sides AB and CD will be equal and opposite, and so will cancel.

The total force is then $\mathbf{F}_{\text{total}} = \mathbf{F}_{DA} + \mathbf{F}_{BC} = (54 - 18)\mathbf{a}_x = 36\,\mathbf{a}_x$ nN

8.

9.7. Uniform current sheets are located in free space as follows: $8\mathbf{a}_z$ A/m at $y = 0$, $-4\mathbf{a}_z$ A/m at $y = 1$, and $-4\mathbf{a}_z$ A/m at $y = -1$. Find the vector force per meter length exerted on a current filament carrying 7 mA in the $\mathbf{a}_L$ direction if the filament is located at:

a) $x = 0$, $y = 0.5$, and $\mathbf{a}_L = \mathbf{a}_z$: We first note that within the region $-1 < y < 1$, the magnetic fields from the two outer sheets (carrying $-4\mathbf{a}_z$ A/m) cancel, leaving only the field from the center sheet. Therefore, $\mathbf{H} = -4\mathbf{a}_x$ A/m $(0 < y < 1)$ and $\mathbf{H} = 4\mathbf{a}_x$ A/m $(-1 < y < 0)$. Outside ($y > 1$ and $y < -1$) the fields from all three sheets cancel, leaving $\mathbf{H} = 0$ ($y > 1$, $y < -1$). So at $x = 0$, $y = .5$, the force per meter length will be

$$\mathbf{F}/m = I\mathbf{a}_z \times \mathbf{B} = (7 \times 10^{-3})\mathbf{a}_z \times -4\mu_0 \mathbf{a}_x = \underline{-35.2\mathbf{a}_y \text{ nN/m}}$$

b.) $y = 0.5$, $z = 0$, and $\mathbf{a}_L = \mathbf{a}_x$: $\mathbf{F}/m = I\mathbf{a}_x \times -4\mu_0\mathbf{a}_x = \underline{0}$.

c) $x = 0$, $y = 1.5$, $\mathbf{a}_L = \mathbf{a}_z$: Since $y = 1.5$, we are in the region in which $\mathbf{B} = 0$, and so the force is zero.

9.

9.18. Calculate the vector torque on the square loop shown in Fig. 9.16 about an origin at A in the field $\mathbf{B}$, given:

a) $A(0,0,0)$ and $\mathbf{B} = 100\mathbf{a}_y$ mT: The field is uniform and so does not produce any translation of the loop. Therefore, we may use $\mathbf{T} = I\mathbf{S} \times \mathbf{B}$ about any origin, where $I = 0.6$ A and $\mathbf{S} = 16\mathbf{a}_z$ m^2. We find $\mathbf{T} = 0.6(16)\mathbf{a}_z \times 0.100\mathbf{a}_y = \underline{-0.96\,\mathbf{a}_x \text{ N}-\text{m}}$.

b) $A(0,0,0)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y$ mT: Using the same reasoning as in part a, we find

$$\mathbf{T} = 0.6(16)\mathbf{a}_z \times (0.200\mathbf{a}_x + 0.100\mathbf{a}_y) = \underline{-0.96\mathbf{a}_x + 1.92\mathbf{a}_y \text{ N}-\text{m}}$$

c) $A(1,2,3)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y - 300\mathbf{a}_z$ mT: We observe two things here: 1) The field is again uniform and so again the torque is independent of the origin chosen, and 2) The field differs from that of part b only by the addition of a z component. With $\mathbf{S}$ in the z direction, this new component of $\mathbf{B}$ will produce no torque, so the answer is the *same as part b*, or $\mathbf{T} = \underline{-0.96\mathbf{a}_x + 1.92\mathbf{a}_y \text{ N}-\text{m}}$.

d) $A(1,2,3)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y - 300\mathbf{a}_z$ mT for $x \geq 2$ and $\mathbf{B} = 0$ elsewhere: Now, force is acting only on the y-directed segment at $x = +2$, so we need to be careful, since translation will occur. So we must use the given origin. The differential torque acting on the differential wire segment at location (2,y) is $d\mathbf{T} = \mathbf{R}(y) \times d\mathbf{F}$, where

$$d\mathbf{F} = I\,d\mathbf{L} \times \mathbf{B} = 0.6\,dy\,\mathbf{a}_y \times [0.2\mathbf{a}_x + 0.1\mathbf{a}_y - 0.3\mathbf{a}_z] = [-0.18\mathbf{a}_x - 0.12\mathbf{a}_z]\,dy$$

and $\mathbf{R}(y) = (2, y, 0) - (1, 2, 3) = \mathbf{a}_x + (y-2)\mathbf{a}_y - 3\mathbf{a}_z$. We thus find

$$\begin{aligned} d\mathbf{T} = \mathbf{R}(y) \times d\mathbf{F} &= [\mathbf{a}_x + (y-2)\mathbf{a}_y - 3\mathbf{a}_z] \times [-0.18\mathbf{a}_x - 0.12\mathbf{a}_z]\,dy \\ &= [-0.12(y-2)\mathbf{a}_x + 0.66\mathbf{a}_y + 0.18(y-2)\mathbf{a}_z]\,dy \end{aligned}$$

The net torque is now

$$\mathbf{T} = \int_{-2}^{2} [-0.12(y-2)\mathbf{a}_x + 0.66\mathbf{a}_y + 0.18(y-2)\mathbf{a}_z]\,dy = \underline{0.96\mathbf{a}_x + 2.64\mathbf{a}_y - 1.44\mathbf{a}_z \text{ N}-\text{m}}$$

10.

9.23. Calculate values for H_ϕ, B_ϕ, and M_ϕ at $\rho = c$ for a coaxial cable with $a = 2.5$ mm and $b = 6$ mm if it carries current $I = 12$ A in the center conductor, and $\mu = 3\ \mu$H/m for $2.5 < \rho < 3.5$ mm, $\mu = 5\ \mu$H/m for $3.5 < \rho < 4.5$ mm, and $\mu = 10\ \mu$H/m for $4.5 < \rho < 6$ mm. Compute for:

a) $c = 3$ mm: Have

$$H_\phi = \frac{I}{2\pi\rho} = \frac{12}{2\pi(3 \times 10^{-3})} = \underline{637 \text{ A/m}}$$

Then $B_\phi = \mu H_\phi = (3 \times 10^{-6})(637) = \underline{1.91 \times 10^{-3} \text{ Wb/m}^2}$.
Finally, $M_\phi = (1/\mu_0)B_\phi - H_\phi = \underline{884 \text{ A/m}}$.

11.

9.27. Let $\mu_{r1} = 2$ in region 1, defined by $2x + 3y - 4z > 1$, while $\mu_{r2} = 5$ in region 2 where $2x + 3y - 4z < 1$. In region 1, $\mathbf{H}_1 = 50\mathbf{a}_x - 30\mathbf{a}_y + 20\mathbf{a}_z$ A/m. Find:

a) $\mathbf{H}_{N1}$ (normal component of $\mathbf{H}_1$ at the boundary): We first need a unit vector normal to the

the gradient of surface $f(x,y,z) = 2x + 3y - 4z$. Then, the normal component of **H** is simply the dot product of **H** with this unit normal multiplied by the unit normal with the answer being $\underline{-4.83\mathbf{a}_x - 7.24\mathbf{a}_y + 9.66\mathbf{a}_z \text{ A/m}}$

(b) the tangential component of **H** is simply the difference between **H** and its normal component.