Antennas – Engineering 7811

Course Notes

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Unit 1

Electromagnetics Review (Mostly)

1.1 Introduction

Antennas act as transducers associated with the region of transition between guided wave structures and free space, or vice versa. The guiding structure could be, for example, a two-wire transmission line or a waveguide (hollow "pipe") leading from a transmitter or receiver to the antenna itself. Generally, the antennas are made of good-conducting material and are designed to have dimensions and shape conducive to radiating or receiving electromagnetic (e-m) energy in an efficient manner. As we shall see, the antenna structure may take many different forms: eg., wires, horns, slots, microstrips, reflectors, and combinations of these.

While we shall be able to examine many important basic characteristics using mathematics appearing earlier in the programme, in most practical situations antenna design must be carried out using sophisticated software – i.e., efficient numerical techniques and packages (such as the Numerical Electromagnetics Code (NEC)) must be employed. The main purpose of this course is to introduce the basics so that future exposure to the engineering software will be meaningful. For the most part, we will not use such software.

1.2 Maxwell's Equations and Related Formulae

Maxwell's Equations

We have previously seen that to properly describe any *time-varying* electromagnetic phenomenon, the following may be invoked (in "point" form):

$$\vec{\nabla} \times \underset{\sim}{\vec{E}} = -\frac{\partial \underset{\sim}{\vec{B}}}{\partial t} \quad (1.1) \quad ; \quad \vec{\nabla} \cdot \underset{\sim}{\vec{D}} = \underset{\sim}{\rho_v} \quad (1.3)$$

$$\vec{\nabla} \times \underset{\sim}{\vec{H}} = \underset{\sim}{\vec{J}} + \frac{\partial \underset{\sim}{\vec{D}}}{\partial t} \quad (1.2) \quad ; \quad \vec{\nabla} \cdot \underset{\sim}{\vec{B}} = 0 \quad (1.4)$$

where $\underset{\sim}{}$ has been used to represent any time variation. Equations (1.1)–(1.4) are referred to as Maxwell's equations. Of course,

$\underset{\sim}{\vec{E}} \equiv \vec{E}(\vec{r}, t) \equiv$ Electric field intensity in V/m.

$\underset{\sim}{\vec{H}} \equiv \vec{H}(\vec{r}, t) \equiv$ Magnetic Field intensity in A/m.

$\underset{\sim}{\vec{D}} \equiv \vec{D}(\vec{r}, t) \equiv$ Electric flux density in C/m^2.

$\underset{\sim}{\vec{B}} \equiv \vec{B}(\vec{r}, t) \equiv$ Magnetic flux density in Wb/m^2 or T (tesla).

$\vec{r} \equiv$ position vector as measured from some origin to a "field" or observation point.

$t \equiv$ time.

$\underset{\sim}{\rho_v} \equiv$ charge density in C/m^3. $\underset{\sim}{\vec{J}} \equiv \vec{J}(\vec{r}, t) \equiv$ current density in A/m^2.

The last two quantities are source terms or "supports" for the field quantities $\underset{\sim}{\vec{E}}$, $\underset{\sim}{\vec{H}}$, $\underset{\sim}{\vec{D}}$, and $\underset{\sim}{\vec{B}}$.

Equation (1.1) is Faraday's law; equation (1.2) is a modification of Ampère's law with $\frac{\partial \underset{\sim}{\vec{D}}}{\partial t}$ being the so-called *displacement current density* (in A/m^2 of course); equation (1.3) is Gauss' law (electric); and equation (1.4) is Gauss' law (magnetic) and it precludes the possibility of magnetic monopoles – i.e., the $\underset{\sim}{\vec{B}}$ field lines do not terminate on a "magnetic charge."

Constitutive Relationships

In addition to Maxwell's equations, we have the following constitutive relationships:

$$\underset{\sim}{\vec{D}} = \epsilon \underset{\sim}{\vec{E}} \quad (1.5) \quad ; \quad \underset{\sim}{\vec{B}} = \mu \underset{\sim}{\vec{H}} \quad (1.6)$$

where ϵ, measured in F/m, is referred to as the *permittivity* of the medium in which the field exists, and μ, in H/m, is the *permeability* of the medium. For homogeneous, isotropic media, ϵ and μ are simply scalars. In our problems, this will always be true (at least, it will be considered to be true). Whenever free space is being considered, ϵ and μ take on the special notation and values given as

$$\epsilon_0 = 8.854 \times 10^{-12} \approx \frac{10^{-9}}{36\pi} \text{ F/m}$$

and

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m}.$$

In general, $\epsilon = \epsilon_0(1 + \chi_e)$ and $\mu = \mu_0(1 + \chi_m)$ where χ_e and χ_m are the electric and magnetic susceptibility, respectively. The former was encountered in Term 5 and the latter in Term 6.

Using the constitutive relationships and the forms of ϵ and μ as given, equations (1.5) and (1.6) become

$$\underset{\sim}{\vec{D}} = \epsilon_0 \underset{\sim}{\vec{E}} + \epsilon_0 \chi_e \underset{\sim}{\vec{E}} = \epsilon_0 \underset{\sim}{\vec{E}} + \underset{\sim}{\vec{P}}$$

and

$$\underset{\sim}{\vec{B}} = \mu_0 \underset{\sim}{\vec{H}} + \mu_0 \chi_m \underset{\sim}{\vec{H}} = \mu_0(\underset{\sim}{\vec{H}} + \underset{\sim}{\vec{M}})$$

where $\underset{\sim}{\vec{P}}$ is called the polarization (due to bound *charges*) and $\underset{\sim}{\vec{M}}$ is called the magnetization (due to bound currents). Recall also, the notion of
(1) *relative* permittivity: $\boxed{\epsilon_R = \epsilon/\epsilon_0}$ and (2) *relative* permeability: $\boxed{\mu_R = \mu/\mu_0}$.

Miscellaneous Relations

Besides equations (1.1) to (1.6), we have the following useful results:

Continuity of Current (or Conservation of Charge):

$$\vec{\nabla} \cdot \underset{\sim}{\vec{J}} = -\frac{\partial \underset{\sim}{\rho_v}}{\partial t}. \tag{1.7}$$

Convection Current Density:

$$\underset{\sim}{\vec{J}}_{cnv} = \underset{\sim}{\rho_v} \vec{v} \tag{1.8}$$

where $\vec{v}$ is the charge velocity.

Ohm's Law:

$$\underset{\sim}{\vec{J}} = \sigma \underset{\sim}{\vec{E}} \tag{1.9}$$

where σ is conductivity in ℧/m.

Lorentz Force Equation:

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B}) \tag{1.10}$$

where Q is charge in coulombs and $\vec{F}$ is force in newtons.

Some Propagation Parameters

Phase Velocity, v_p: In general,

$$v_p = \frac{1}{\sqrt{\mu\epsilon}} \tag{1.11}$$

or for free space

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} \tag{1.12}$$

where $c = 3 \times 10^8$ m/s. Also,

$$v_p = f\lambda \tag{1.13}$$

where f is the frequency in hertz (Hz) and λ is wavelength in metres.

Wave Number and Wave Vector: We define a wave number k for lossless media as

$$k = \frac{2\pi}{\lambda} = \omega\sqrt{\mu\epsilon} \tag{1.14}$$

where k is in radians per metre and the radian frequency $\omega = 2\pi f$. In the case of the lossless medium, k is equivalent to the β of the Term 6 course.

If the medium is lossy, k may be complex and we define

$$jk = \alpha + j\beta\ . \tag{1.15}$$

In this case, k is referred to as the *complex propagation constant* and jk is the same as γ of the Term 6 course. The quantity α is the attenuation coefficient in nepers per metre (Np/m). In the lossy case,

$$\beta = \frac{2\pi}{\lambda} \, .$$

For plane wave propagation in lossless isotropic media, we may define a *wave vector*, $\vec{k}$, such that

$$k = |\vec{k}| = \frac{2\pi}{\lambda}$$

and the direction of $\vec{k}$ is the direction of wave energy flow.

Intrinsic Impedance: Finally, we define the *intrinsic impedance*, η, (in ohms) of a medium in which a wavefield exists as

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \tag{1.16}$$

which for free space becomes

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \tag{1.17}$$

It may be noted that $\eta_0 \approx 120\pi\ \Omega$.

Time-Harmonic Fields

We discovered previously that one solution to Maxwell's equations was indicative of plane waves travelling through a medium whose properties are stipulated by the values of ϵ, μ, σ, and so on. In arriving at this conclusion, we started with time-harmonic (sinusoidal) fields. Recall, for a time harmonic field, $\utilde{\vec{A}}$, of the form $\utilde{\vec{A}} = \vec{A}_0 \cos(\omega t + \phi)$, that

$$\utilde{\vec{A}}(\vec{r}, t) \equiv \utilde{\vec{A}} = \mathcal{R}e \left\{ \vec{A}_s e^{j\omega t} \right\} \tag{1.18}$$

where $\vec{A}_s$ is the phasor form of the field. Also, recall that the time derivative $\partial/\partial t$ transforms to $j\omega$ in the phasor domain. From now on, since it is generally to be understood almost everywhere IN THIS COURSE that the fields are time-harmonic, we shall drop the s subscript on the phasor and use simply the form $\vec{A}$. Equations

(1.1)–(1.4) in phasor form become

$$\vec{\nabla} \times \vec{E} = -j\omega\vec{B} \quad (1.19) \;\; ; \;\; \vec{\nabla} \cdot \vec{D} = \rho_v \quad (1.21)$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + j\omega\vec{D} \quad (1.20) \;\; ; \;\; \vec{\nabla} \cdot \vec{B} = 0 \quad (1.22)$$

In these equations, for free space, $\vec{J} = 0$ and $\rho_v = 0$.

(ASIDE: Many texts use $\tilde{E}$, etc. to denote phasors).

Boundary Conditions

There are many instances in which e-m energy impinges a boundary between two electromagnetically distinct media. For example,

In general, for two media as shown, where $\hat{n}$ is the unit normal to the boundary or interface,

the following important relationships hold:

(1) The tangential $\vec{E}$-field is continuous across the boundary; i.e.,

$$\hat{n} \times \vec{E}_1 = \hat{n} \times \vec{E}_2 \; .$$

(What does this imply if medium 2 is a perfect conductor?)

(2) If no surface current density ($\vec{K}$) exists on the boundary, then the tangential $\vec{H}$-field is continuous across the boundary i.e.,

$$\hat{n} \times \vec{H}_1 = \hat{n} \times \vec{H}_2 \; ;$$

else the tangential $\vec{H}$-field is discontinuous by an amount equal to $\vec{K}$; i.e.,

$$\hat{n} \times \left[\vec{H}_1 - \vec{H}_2\right] = \vec{K} \ .$$

(3) The normal component of the $\vec{D}$-field is discontinuous by an amount equal to the surface charge density, ρ_s, on the boundary; i.e.,

$$\hat{n} \cdot \left[\vec{D}_1 - \vec{D}_2\right] = \rho_s \ .$$

(What's the implication if medium 2 is a perfect conductor? The answer can also be arrived at by applying Gauss' law.)

(4) The normal component of the $\vec{B}$-field is continuous across the boundary; i.e.,

$$\hat{n} \cdot \vec{B}_1 = \hat{n} \cdot \vec{B}_2 \ .$$

1.3 Scalar and Vector Potentials

Review

In earlier electromagnetics courses, it was observed that implementing constructs referred to as "potentials" facilitated the calculation of the $\vec{E}$- and $\vec{B}$-field quantities. In general, it was found that the "potential" integrals were easier to calculate than the field expressions appearing, for example, in Coulomb's law (static electric field) or in the Biot-Savart law (steady magnetic field). Furthermore, on determining the potentials, the $\vec{E}$ and $\vec{B}$ fields were readily found using derivatives rather than integrals.

Recall the following geometry for the specification of a field point $P(x, y, z)$:

$\vec{r} \equiv$ position vector for *observation or field* point, P.

$\vec{r}' \equiv$ position vector for points in the *source* region.

$\vec{r} - \vec{r}'$ is as shown.

Electrostatic Field: The source consists of electric charges. It was discovered that if a scalar potential, say Φ, existed at P, then the electric field was simply given by

$$\vec{E}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}) \tag{1.23}$$

Note that Φ is the V of Term 6 and

$$\Phi(\vec{r}) = V(\vec{r}) = \int_{v'} \frac{\rho_v(\vec{r}')dv'}{4\pi\epsilon|\vec{r} - \vec{r}'|} \tag{1.24}$$

where (in Cartesian coordinates), $\int_{v'} \ldots dv' \equiv \int_{z'} \int_{y'} \int_{x'} \ldots dx'dy'dz'$ and ρ_v is the charge density. Equation (1.24) is the solution to Poisson's equation

$$\vec{\nabla}^2\Phi = -\frac{\rho_v}{\epsilon} \tag{1.25}$$

Note that equation (1.24) could be also written explicitly as a 3-dimensional spatial convolution:

$$\begin{aligned} \Phi(x, y, z) &= \int_{z'} \int_{y'} \int_{x'} \frac{\rho_v(x', y', z')dx'dy'dz'}{4\pi\epsilon\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \\ &= \frac{\rho_v(x, y, z)}{\epsilon} \overset{*}{_{3d}} \frac{1}{4\pi|\vec{r}|} \end{aligned} \tag{1.26}$$

with $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ while $\overset{*}{3d}$ represents a 3-dimensional convolution.

Magnetostatic Field: The source consists of a steady current. We have seen that steady currents (i.e., dc) produce steady magnetic fields. In our analysis we introduced a "vector" potential, $\vec{A}$, defined as

$$\vec{B} = \vec{\nabla} \times \vec{A} \, . \tag{1.27}$$

Making the substitution into Maxwell's equations and invoking the Coulomb gauge ($\vec{\nabla} \cdot \vec{A}$=0) – which we proved had to be true for steady fields as a result of there being no $(\partial/\partial t)$ terms – it was shown that

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J} \, . \tag{1.28}$$

On comparing (1.28) with (1.25), while keeping an eye on (1.24) and (1.26), we may immediately write that

$$\begin{aligned} \vec{A}(\vec{r}) &= \int_{v'} \frac{\mu_0 \vec{J}(\vec{r'}) dv'}{4\pi |\vec{r} - \vec{r'}|} \\ &= \mu_0 \vec{J}(x, y, z) \overset{*}{_{3d}} \frac{1}{4\pi |\vec{r}|} \end{aligned} \tag{1.29}$$

Generally, equation (1.27) and (1.29) together give a simpler way of calculating the magnetic flux density, $\vec{B}$, (or, equivalently, the magnetic field intensity, $\vec{H}$) than is available via the Biot-Savart law which contains a cross product.

New Material on Potentials

So far, we have considered only the non-time-varying field. However, in antenna theory, the fields are time varying. In fact, as we shall elaborate, the production of radiated energy requires charges to be *accelerating*. Therefore, the steady field forms for the vector and scalar potentials must be revisited for this new case. We shall assume time-harmonic sources and fields.

From equation (1.19),

(Since "the curl of the gradient" is always zero). However, if $\vec{\nabla} \times \vec{E} = 0$, there cannot be a time-varying $\vec{B}$-field (see equation (1.1) and remember that the curl is a spatial operator). Since the discussion has now turned to time-varying fields it is clear that our "old" (i.e., steady field) scalar potential cannot be used. The details are a little more complicated this time, but the starting point is our observation that a time-varying $\vec{B}$-field is still *solenoidal* or *divergenceless*: i.e.,

Therefore, a *vector* potential, $\vec{A}$, is still generally defined by (1.27)

$$\vec{B} = \vec{\nabla} \times \vec{A} \, .$$

However, we cannot invoke the Coulomb gauge ($\vec{\nabla} \cdot \vec{A} = 0$) and expect $\vec{A}$ to be useful in determining time-varying fields. What to do?!! Stay tuned (if you are still tuned)!

Using equation (1.27) in

$$\vec{\nabla} \times \vec{E} = -j\omega \vec{B} \, ,$$

we get

which implies

(1.30)

Since "the curl of a gradient is zero", equation (1.30) implies that

(1.31)

where Φ is a *scalar* potential. (Note that the "$-$" allows us to write $\vec{E} = -\vec{\nabla}\Phi$ when $\vec{A}$ is not time-varying: recall $j\omega \leftrightarrow \frac{\partial}{\partial t}$ and $\frac{\partial \vec{A}}{\partial t} = 0$ when $\vec{A}$ has no time dependence.) For time-harmonic fields, equation (1.31) gives

(1.32)

and it is seen that the $\vec{E}$ field depends on both the scalar and vector potential. Now, (1.27) and (1.32) satisfy

automatically, since that's how we started. What about the remaining Maxwell equations?

$$(1.33)$$

$$(1.34)$$

Since (1.33) and (1.34) must also be satisfied by (1.27) and (1.32), it looks like picking the proper forms of $\vec{A}$ and Φ could be quite messy! Substituting from equations (1.27) and (1.32) into (1.33) gives

$$(1.35)$$

Invoking the vector identity $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$, equation (1.35) becomes

Recalling $k = \omega\sqrt{\mu\epsilon}$,

$$\vec{\nabla}^2 \vec{A} + k^2 \vec{A} = \vec{\nabla}\left\{(\vec{\nabla} \cdot \vec{A}) + j\omega\mu\epsilon\Phi\right\} - \mu\vec{J}\,. \tag{1.36}$$

We have argued in Term 6 that to completely specify a vector, both the curl and divergence are required. (It may be seen that the curl alone is not enough to uniquely define $\vec{A}$ since $\vec{\nabla} \times (\vec{A} + \vec{\nabla}\lambda) = \vec{\nabla} \times \vec{A}$ for any scalar function λ). Of course, the curl is specified here by $\vec{\nabla} \times \vec{A} = \vec{B}$..... but what are we to do about the divergence $(\vec{\nabla} \cdot \vec{A})$ in (1.36)? In electro/magnetostatics we invoked the Coulomb gauge, which we have said is not a good plan for time-varying fields. With a view to eliminating the gradient on the R.H.S. of (1.36), we DEFINE AND USE the *Lorenz* gauge:

$$\vec{\nabla} \cdot \vec{A} = -j\omega\mu\epsilon\Phi \tag{1.37}$$

This is a good choice as, in retrospect, it is seen to lead to all of the proper results for the fields.

On using the Lorenz gauge, equation (1.36) becomes

$$\vec{\nabla}^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J} \tag{1.38}$$

and from (1.32), (1.34) and (1.37)

$$\vec{\nabla}^2 \Phi + k^2 \Phi = -\frac{\rho_v}{\epsilon} \tag{1.39}$$

Equations (1.38) and (1.39) are the *inhomogeneous* Helmoltz equations for vector and scalar potentials. We note that for the static case, in which $k = 0$ since $\omega = 0$, (1.38) and (1.39) reduce to the proper forms given by (1.25) and (1.28). It is possible to develop solutions to (1.38) and (1.39) by analogy to the static case (of course, they may be solved rigorously also). We will not give a rigorous solution in this course. Rather, let's write down the "answers" and observe some of the properties which seem to make intuitive sense. The answers are very important because they eventually lead to the time-varying fields produced by time-harmonic sources. We get

$$\vec{A}(\vec{r}) = \int_{v'} \frac{\mu_0 \vec{J}(\vec{r'}) e^{-jk|\vec{r}-\vec{r'}|} dv'}{4\pi |\vec{r} - \vec{r'}|} \tag{1.40}$$

and

$$\Phi(\vec{r}) = \int_{v'} \frac{\rho_v(\vec{r'}) e^{-jk|\vec{r}-\vec{r'}|} dv'}{4\pi\epsilon |\vec{r} - \vec{r'}|} \tag{1.41}$$

Since $\vec{E}$ and $\vec{B}$ can be determined from $\vec{A}$ via equations (1.27), (1.33), and (1.38), let's consider (1.40) in detail:

<u>Observations of Similarities Between (1.29) and (1.40):</u>

1. Mathematically, at points removed from the source (i.e., $\vec{r} \neq \vec{r'}$), the static and time-varying forms differ by the presence of a phase term, $e^{-jk|\vec{r}-\vec{r'}|}$ in the latter. (Note that this is similarly the case for the Φ's of equations (1.26) and (1.41)). That is, physically, as compared to the static case, the $\vec{A}$ viewed at

the field position $\vec{r}$ is *phase-delayed* by an amount determined by the distance $R = |\vec{r} - \vec{r}'|$ as shown:

That is, the phase delay is determined by the distance from the source to the observer.

2. If we take (1.40) to the time domain, we would see that $\underset{\sim}{\vec{A}}$ at $\vec{r}$ is determined by the state of the source at time (R/c) seconds earlier. Of course, (R/c) is simply the time necessary for a "phenomenon" travelling at the speed of light to cover the the distance R. For this reason the potentials in (1.40) and (1.41) are referred to as "retarded" potentials.

3. The amplitude of $\vec{A}$ decreases as $1/R$.

We have, then, the following procedure for finding the $\vec{E}$ and $\vec{B}$ fields due to a source:

It may be noted that $\vec{J}dv'$ could be replaced by $\vec{K}dS'$ or $Id\ell'\hat{\ell}'$ for surface and line currents, respectively, and the triple integral in $\vec{A}$ would accordingly reduce to a double or single integral. The important point is that $\vec{A}$ is derived from the current source, no matter what that source might be.

We are finally in a position to begin our discussion of radiated energy due to time-varying sources. In what follows, the "sources" are antenna currents.