Unit 3

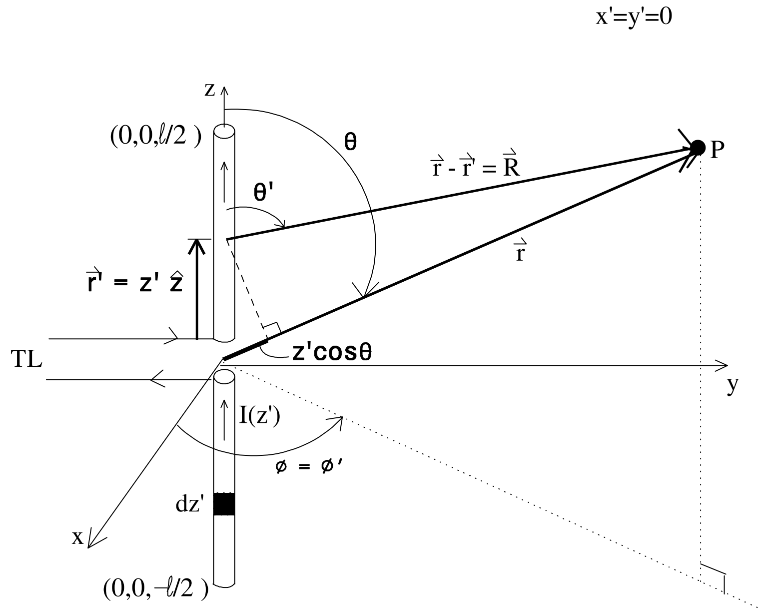
Linear Wire Antennas and Antenna Arrays

While the poor efficiency of the "small" antennas discussed in the last unit limits their practicality, the ideas encountered in analyzing them are very useful in establishing the radiation characteristics of larger, more efficient, structures. The main restriction in what follows is the imposition of the requirement that the linear wires involved have a diameter (d) which is small compared to a wavelength (λ) – i.e., they are *thin* antennas. If $d \ll \lambda$, the current along the arbitrarily long dipole is essentially sinsoidal with nulls at each end (more on this shortly) and the analysis is fairly straightforward. The case of $d \not\ll \lambda$ is not considered in this course.

A linear antenna may be characterized as an arbitrary current on a thin straight conductor lying along the z-axis and centred at $(0, 0, 0)$. This structure is fed at its midpoint by a transmission line – i.e., this is a dipole antenna with length, ℓ, unrestricted. The two "arms" of the dipole are the regions where curents can flow and charges accumulate.

3.1 Thin, Linear Wire Dipole Antennas

Consider a thin dipole antenna of arbitrary length being fed at its centre as shown below. The antenna is chosen for the purpose of this analysis to lie along the z-axis and to be centred at $(0, 0, 0)$. We shall examine the far field of this structure.



Facts: (1.) The dipole is thin and the current is essentially sinusoidal – we shall use the form $I(z')$ as the phasor current.

(2.) The current goes to zero at the ends of the dipole – i.e., current nulls exist at the ends.

(3.) The current is given by

$$I(z') = I_0 \sin\left[k\left(\frac{\ell}{2} - |z'|\right)\right] \tag{3.1}$$

where $k = 2\pi/\lambda$ as before. It may be verified experimentally – as well as from a variety of theoretical methods – that equation (3.1) is an extremely good representation of the current on thin linear dipoles. The expression $\sin\left[k\left(\frac{\ell}{2} - |z'|\right)\right]$ is referred to as the *form factor* for the current on the antenna. Here, then, equation (1) is taken as a "given".

3.1.1 The Vector Potential and Resulting Fields of the Thin Linear Dipole

The usual approach of determining the potential, $\vec{A}(\vec{r})$, in order to find the $\vec{E}$ and $\vec{H}$ fields will again be followed. As previously, we have for the vector potential, in general for a line current,

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi}\hat{z}\int_{-\ell/2}^{\ell/2} \frac{I(z')e^{-jk|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}dz' \tag{3.2}$$

Because the dipole is "thin" and lies along the z-axis, $x' = y' = 0$ and

$$|\vec{r}-\vec{r}'| = \sqrt{x^2 + y^2 + (z-z')^2}\ . \tag{3.3}$$

For the far-field, we have shown in Section 2.2.2 that, since $r \gg z'$, equation (3.3) reduces to

$$|\vec{r}-\vec{r}'| \approx r - z'\cos\theta\ . \tag{3.4}$$

Using (3.4) in the phase factor and $|\vec{r}-\vec{r}'| \approx r$ in the amplitude factor of (3.2), we write

(3.5)

Putting the expression for the dipole current of equation (3.1) into (3.5) yields

(3.6)

Doing the integration, it is easily verified that

$$\vec{A}(\vec{r}) = \hat{z}\frac{\mu I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{k\ell}{2}\cos\theta\right) - \cos\left(\frac{k\ell}{2}\right)}{k\sin^2\theta}\right]\ . \tag{3.7}$$

Beginning with $\vec{H} = \frac{1}{\mu}\vec{\nabla}\times\vec{A}$, it is straightforward to determine that in the far-field (where we retain terms $\propto \frac{1}{r}$ and drop higher orders of $\frac{1}{r}$)

$$\vec{E}(\vec{r}) = E_\theta\hat{\theta} = \frac{j\eta I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{k\ell}{2}\cos\theta\right) - \cos\left(\frac{k\ell}{2}\right)}{\sin\theta}\right]\hat{\theta} \tag{3.8}$$

and

$$\vec{H}(\vec{r}) = H_\phi\hat{\phi} = \frac{jI_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{k\ell}{2}\cos\theta\right) - \cos\left(\frac{k\ell}{2}\right)}{\sin\theta}\right]\hat{\phi}\ . \tag{3.9}$$

The magnitudes of the expressions in the square brackets are referred to as the E- and H-field pattern (or amplitude pattern), $f(\theta)$, respectively (or, in general, $f(\theta,\phi)$).

3.1.2 Power Density and Other Parameters

The other far-field characteristics, such as the Poynting vector, radiation intensity, radiation resistance and so on may now be determined from equations (3.8) and (3.9).

Time-Averaged Poynting Vector, $\vec{\mathcal{P}}_a$:

The time-averaged Poynting vector, $\vec{\mathcal{P}}_a$ may be determined as usual according to

$$\vec{\mathcal{P}}_a =$$

$$=$$

$$\Rightarrow \vec{\mathcal{P}}_a = \frac{\eta |I_0|^2}{8\pi^2 r^2} \left[\frac{\cos\left(\frac{k\ell}{2}\cos\theta\right) - \cos\left(\frac{k\ell}{2}\right)}{\sin\theta} \right]^2 \hat{r} \ \ \mathrm{W/m^2} \ . \tag{3.10}$$

Radiation Intensity, $U \equiv U(\theta, \phi)$:

The radiation intensity, U may be determined as usual according to

Therefore,

$$U = \frac{\eta |I_0|^2}{8\pi^2} \left[\frac{\cos\left(\frac{k\ell}{2}\cos\theta\right) - \cos\left(\frac{k\ell}{2}\right)}{\sin\theta} \right]^2 \ \ \mathrm{W/sr} \tag{3.11}$$

Average Radiated Power, P_r, and Radiation Resistance, R_r:

Again, considering the power density over a sphere surrounding the dipole we get as before (see equation (2.33))

$$P_r = \frac{\eta |I_0|^2}{4\pi} \int_0^{\pi} \frac{\left[\cos\left(\frac{k\ell}{2}\cos\theta\right) - \cos\left(\frac{k\ell}{2}\right)\right]^2}{\sin\theta} d\theta \ \ \mathrm{W} \ . \tag{3.12}$$

This is the general form of the *radiated* power from a dipole of arbitrary length. This final integral is not "simple" and it doesn't reduce to a closed-form result. After four or five pages of tedious manipulations, it transpires that

$$
\begin{aligned}
P_r &= \frac{\eta |I_0|^2}{4\pi} \left\{ C + \ln(k\ell) - C_i(k\ell) + \frac{1}{2}\sin(k\ell)\left[S_i(2k\ell) - 2S_i(k\ell)\right] \right. \\
&\quad \left. + \frac{1}{2}\cos(k\ell)\left[C + \ln\left(\frac{k\ell}{2}\right) + C_i(2k\ell) - 2C_i(k\ell)\right]\right\} \; W \qquad (3.13)
\end{aligned}
$$

where $C = 0.5772$ is called Euler's constant and the Cosine and Sine integrals are defined as follows:

$$C_i(x) = -\int_x^\infty \frac{\cos y}{y} dy = \int_\infty^x \frac{\cos y}{y} dy \quad \text{(Cosine Integral)}$$

$$S_i(x) = \int_0^x \frac{\sin y}{y} dy \quad \text{(Sine Integral)}$$

These integrals do not exist in closed form, but they may be "performed" as series expansions. Of course, with numerical integration (eg., Matlab, Mathematica, Maple, etc.) numerical answers may usually be obtained very efficiently.

From equation (2.28) and (3.13), the radiation resistance is given by

$$
\begin{aligned}
R_r = \frac{2P_r}{|I_0|^2} &= \frac{\eta}{2\pi} \left\{ C + \ln(k\ell) - Ci(k\ell) + \frac{1}{2}\sin(k\ell)\left[Si(2k\ell) - 2Si(k\ell)\right] \right. \\
&\quad \left. + \frac{1}{2}\cos(k\ell)\left[C + \ln\left(\frac{k\ell}{2}\right) + Ci(2k\ell) - 2Ci(k\ell)\right]\right\} \; \Omega \qquad (3.14)
\end{aligned}
$$

3.1.3 Special Cases of the Thin Dipole

Having the general forms for the important parameters of the thin dipole, we will now consider one special case in detail and briefly refer to some others.

The Half-Wave Dipole

Current Distribution

For the half-wave dipole, by definition, $\ell = \lambda/2$. Substituting this into equation (3.1) yields for the current distribution

$$
\begin{aligned}
I(z') &= \\
&=
\end{aligned}
$$

$$\Rightarrow I(z') = I_0 \cos(k|z'|) \tag{3.15}$$

$\vec{E}$ and $\vec{H}$ Fields:

Noting in (3.8) and (3.9) that $k\ell/2 = (2\pi\lambda)/(\lambda \cdot 2 \cdot 2) = \pi/2$,

$$\tag{3.16}$$

$$\tag{3.17}$$

Clearly, the field patterns (i.e. the amplitude patterns) are given by

$$\left| f(\theta)|_{\ell=\lambda/2} \right| = \left| \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right| \tag{3.18}$$

The field pattern may be easily established numerically (or may be checked analytically for maxima and minima):

$$\tag{3.19}$$

Setting $f'(\theta) = 0$ to seek maxima and minima eventually leads to

$$\frac{\pi}{2}\sin\left(\frac{\pi}{2}\cos\theta\right)\sin^2\theta = \cos\theta\cos\left(\frac{\pi}{2}\cos\theta\right)$$

provided we put the restriction $\sin\theta \neq 0$ (hold on for this case). A quick check (not a proof!) reveals that $\theta = \frac{\pi}{2}$ creates equality in the last expression. From (3.19)

Now, to get the minima in the pattern, we can't (without checking) simply set the numerator to zero in (3.19) because the "solution" is then $\theta = 0, \pi$ and this makes the denominator 0 also – i.e. $f(\theta) \longrightarrow \frac{0}{0}$ which is an indeterminate form. However, using L'Hopital's rule on (3.19) gives

Therefore, there are indeed nulls at $\theta = 0, \pi$. These results should not be surprising in view of the current pattern on the dipole as given above.

Principal Plane Patterns:

Time-Averaged Poynting Vector, $\vec{\mathcal{P}}_{a_{\frac{\lambda}{2}}}$:

Using $\ell = \lambda/2$ and noting that $k\ell/2 = \pi/2$, equation (3.10) readily gives for the half-wave dipole

$$\vec{\mathcal{P}}_{a_{\frac{\lambda}{2}}} = \frac{\eta |I_0|^2}{8\pi^2 r^2} \frac{\cos^2\left(\frac{k\ell}{2}\cos\theta\right)}{\sin^2\theta} \hat{r} \ \ \text{W/m}^2 \ . \tag{3.20}$$

Radiation Intensity, $U_{\frac{\lambda}{2}}$:

Using $\ell = \lambda/2$ and noting that $k\ell/2 = \pi/2$, equation (3.11) readily gives for the half-wave dipole

$$U_{\frac{\lambda}{2}} = r^2 \left|\vec{\mathcal{P}}_{a_{\frac{\lambda}{2}}}\right| = \frac{\eta |I_0|^2}{8\pi^2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \ \ \text{W/sr} \ . \tag{3.21}$$

Note that the power pattern shape is given by

$$F(\theta,\phi) = |f(\theta,\phi)|^2 = \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

<u>Radiated Power, $P_{r_{\frac{\lambda}{2}}}$:</u>

From equations (2.33) and (3.21), it is easily seen that the radiated power is given by

This must be done numerically, in which case it can be shown that the final integral has a value $\int_0^\pi = 1.218$. Then, using the free-space value of $\eta_0 = 120\pi$,

$$P_{r_{\frac{\lambda}{2}}} = 36.54|I_0|^2 \ \ \text{W} \tag{3.22}$$

Note that this result could have been readily obtained by programming equation (3.13).

<u>Radiation Resistance, $R_{r_{\frac{\lambda}{2}}}$:</u>

As usual, set $P_{r_{\frac{\lambda}{2}}} = \frac{1}{2}|I_0|^2 R_{r_{\frac{\lambda}{2}}}$ and from (3.22)

$$R_{r_{\frac{\lambda}{2}}} = 2 \times 36.54 = 73.1\Omega\ .$$

This is the radiation resistance of the thin half-wave dipole in free space.

<u>Current Distributions on Other Dipoles and Relative $\vec{E}$-Field Patterns</u>

Recall $I(z') = I_0 \sin\left[k\left(\frac{\ell}{2} - |z'|\right)\right]$.

<u>$\ell < \frac{\lambda}{2}$:</u>

$\underline{\frac{\lambda}{2} < \ell < \lambda}$:

Up to a limit, the beam becomes more directive, with maximum gain in the horizontal direction, as the antenna length increases – at $\ell > \frac{5\lambda}{4}$ other lobes appear in the pattern; that is, the beam splits up.

$\underline{\lambda < \ell < \frac{3\lambda}{2}}$:

3.2 Images and Monopoles

The concept of "image theory" in electromagnetics states that "any given charge configuration above an infinite, perfectly conducting plane is electrically equivalent to the combination of the the given charge configuration and its image configuration with the conducting plane removed". For this to "work"

1. the *image* charge configuration must be placed in the region (half-space) containing the conductor

   and

2. the image charge configuration must be placed so that the potential at the conductor surface is a constant (generally zero – at infinity the potential is also zero).

Illustration:

Repeating our previous theory for the second situation leads to the same $\vec{E}$, $\vec{H}$, and $\vec{\mathcal{P}}_a$ expressions as before – but only for the $z > 0$ region! The results DO NOT APPLY in the $z < 0$ region. The second illustration above is that of a *monopole* antenna. It could be realized, for example, by extending the centre conductor of a coaxial cable through a ground plane while the outer conductor is attached to the plane.

In the present context of thin wire antennas, consider a quarter-wave ($\lambda/4$) monopole:

While the fields and Poynting vector using the antenna and the image have the same expressions as for the $\lambda/2$ dipole in free space, the radiated power is only half that of the dipole. This is easy to believe since in the $z > 0$ region (region of analysis validity), $0 \leq \theta \leq \frac{\pi}{2}$ while for the dipole in free space, $0 \leq \theta \leq \pi$. Therefore, the following hold:

Note: Here we have considered the antenna to be at the level of the ground plane (not raised above it).

Similar analysis could be carried out for any antenna configuration. For example:

Note, additionally, that if the ground is not perfectly conducting – i.e. the case for all real-life situations – the field patterns will not be exactly as discussed. In fact, significant modifications may exist depending on the electrical characteristics, in particular σ and ϵ, of the ground. Other "real" problems which may need consideration arise from the fact that the ground may not be planar in the the antenna's field of view. There are theoretical analyses as well as practical computer programs which may be used to address such issues. We will not consider them further here.

3.3 Antenna Arrays

We have seen that enlarging the dimensions electrically – which may be done by enlarging the physical dimensions or by electrical means (for example, recall the "capacitive hats" discussed in the context of the small element in Section 2.2.1) – makes an antenna more directive (higher gain). Another way of increasing the directive characteristics is to use multiple elements in special electrical and geometrical configurations. Such a combination of antenna elements is called an *antenna array*. In this section, we shall consider arrays of dipole elements. The nature of the resulting antenna pattern will depend on several controlling factors:

1. the geometrical layout of the array – i.e., the shape of the combination of elements, whether it be linear, circular, rectangular, etc..

2. the relative displacement between the elements – usually referred to in terms of wavelength.

3. the excitation amplitude of the individual elements.

4. the excitation phase of the individual elements.

5. the relative pattern (power or field pattern) of the individual elements.

A very important class of antenna elements is the *linear array* formed by placing identical, equally spaced elements along a line. As well, more complicated analytically, are *planar arrays* and *circular arrays* (a special case of planar arrays) arranged as shown.

We shall begin our analysis with the 2-element array.

3.3.1 Two-Element (Dipole) Array – Free Space Analysis

To simplify the analysis, we consider the following:

1. both elements have the same physical and electrical properties (i.e., length, material, and thus radiation resistance, etc. are the same).

2. there is no coupling between the two elements.

Property (2) is not likely to be satisfied in "real life", but as a reasonable approximation it allows us to get some idea of the resulting pattern without undue mathematical complications. Remember, too, that in actual implementations neither free space nor perfect ground are exactly realizable. For particular implementatioms of an antenna array, the patterns may be measured directly after the initial design and installation. Several numerical (computer) techniques provide an excellent means of giving a good idea of antenna performance before the actual fabrication and installation occurs. With these facts in mind, we proceed with the case of the ideal two-element array in free space and seek the E-field patterns.

First, recall the following examples:

1. Elementary (i.e. Infinitesimal) Dipole Source: To emphasize that, in general, $\vec{E}$ may be a function of r, θ, and ϕ and noting $\eta = 120\pi$ for free space while $k = 2\pi/\lambda$, we write equation (2.16) as

   which implies

$$\vec{E}(r,\theta,\phi) = \frac{CI_0}{r} e^{-jkr} \vec{E}_a(\theta,\phi) \tag{3.23}$$

   where $C = \dfrac{j60\pi\ell}{\lambda}$ is a complex constant and

$$\vec{E}_a(\theta,\phi) = \sin\theta\hat{\theta}$$

   is the direction-dependent factor. Recall that the far-field, principal plane patterns are:

2. Half-Wave Dipole: From equation (3.16) (note that we could use (3.8) and discuss the general dipole)

$$\vec{E}(r,\theta,\phi) = \frac{CI_0}{r} e^{-jkr} \vec{E}_a(\theta,\phi) \tag{3.24}$$

   where, now, $C = j60$ and

$$\vec{E}_a(\theta,\phi) = \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right] \hat{\theta} \ .$$

   The principal plane patterns are

The idea that we get from (3.23) and (3.24) is that the far-field $\vec{E}$ of a vertical dipole antenna may be written in general form as

$$\vec{E}(r,\theta,\phi) = [\text{ Complex Constant }] \times \left[\frac{I_0}{r} e^{-jkr}\right] \times [\text{ Direction Dependent Factor }] \tag{3.25}$$

With this information in place, let us now consider an “array” of two identical vertical dipole antennas. Furthermore, we shall consider only the far field (i.e. $r \gg d$ where d is the distance between the elements). For reference purposes, let’s put the elements along the x-axis as shown (this is totally arbitrary – we could use any axis). Because $r \gg d$, the θ and ϕ coordinates are approximately the same with respect to both elements. We shall label the first antenna 0 and the second, 1.

Element 0 carries current I_0 and element 1 carries current I_1 – this means that I_1 replaces I_0 in the field equation for the element involved. We note the following:

1. From the point of view of magnitude, in the far field, $r_0 \approx r_1$ and, of course,

$$\frac{1}{r_1} \approx \frac{1}{r_0} \qquad \text{FACT 1}$$

2. However, as discussed earlier, we can’t make this approximation in the phase. Rather, from the above diagram,

$$r_1 = r_0 - d\cos\alpha \qquad \text{FACT 2}$$

We’ll use $\vec{E}_0(r_0,\theta,\phi)$ and $\vec{E}_1(r_1,\theta,\phi)$ for the far fields of elements 0 and 1 respectively, with corresponding direction-dependent parts of $\vec{E}_{0a}$ and $\vec{E}_{1a}$ as in equations

(3.23) or (3.24) or (3.25). Then, for the first element

$$\vec{E}_0(r_0, \theta, \phi) = \frac{CI_0}{r_0} e^{-jkr_0} \vec{E}_{0a}(\theta, \phi) \tag{3.26}$$

and for the second element

$$\vec{E}_1(r_1, \theta, \phi) = \frac{CI_1}{r_1} e^{-jkr_1} \vec{E}_{1a}(\theta, \phi)$$

or, because the direction characteristics of identical elements are identical,

$$\vec{E}_1(r_1, \theta, \phi) = \frac{CI_1}{r_1} e^{-jkr_1} \vec{E}_{0a}(\theta, \phi) \tag{3.27}$$

Now, the total $\vec{E}$-field, $\vec{E}_T$, may be written as

(3.28)

on using (3.26) and (3.27) in (3.28) along with FACTS 1 and 2,

$$\vec{E}_T(r_0, \theta, \phi) = C \left[\frac{I_0 e^{-jkr_0}}{r_0} + \frac{I_1 e^{-jk(r_0 - d\cos\alpha)}}{r_0} \right] \vec{E}_{0a}(\theta, \phi) \tag{3.29}$$

Let's further assume that I_1 is related to I_0 via

where $A = \dfrac{|I_1|}{|I_0|}$ and β is the phase difference between the two currents – i.e. we are allowing for scalar multiplication in amplitude and a phase shift. Clearly, (3.29) becomes

$$\vec{E}_T = \left[\frac{CI_0 e^{-jkr_0}}{r_0} \vec{E}_{0a}(\theta, \phi) \right] \left[1 + Ae^{j(kd\cos\alpha + \beta)} \right]$$

or

$$\vec{E}_T = \vec{E}_0(r_0, \theta, \phi) \left[1 + Ae^{jkd\cos\alpha + j\beta} \right] \tag{3.30}$$

<u>Important note:</u>

Pattern Multiplication

Let's write equation (3.30) in the form

$$\vec{E}_T = \vec{E}_0(r_0, \theta, \phi) E'(\alpha, \beta) \tag{3.31}$$

where $E'(\alpha, \beta) = (1 + Ae^{(jkd\cos\alpha + j\beta)})$ is called the *array factor* and $\vec{E}_0(r_0, \theta, \phi)$ is the field pattern for an individual element. We have seen that the general shape of the $\vec{E}_0(r_0, \theta, \phi)$ pattern in both the horizontal and vertical planes for vertical dipoles on several different occasions already.

Now, the pattern based on the array factor is called the *group pattern* and is found by plotting the magnitude $|E'(\alpha, \beta)|$, while bearing in mind the earlier definition of α. Thus, based on (3.30) and (3.31), we see that

$$\boxed{\text{Resultant Pattern} = \text{Unit Pattern} \times \text{Group Pattern}} \tag{3.32}$$

and this may be readily deduced for either the horizontal or vertical planes – i.e., azimuth and elevation.

Special Case $|I_1| = |I_0|$

In this case, $A = 1$ in equation (3.30). Then,

$$|E'(\alpha, \beta)| = \left|1 + e^{(jkd\cos\alpha + j\beta)}\right| \ ,$$

which is easily shown to be (do this)

$$|E'(\alpha, \beta)| = 2\left|\cos\left(\frac{kd\cos\alpha + \beta}{2}\right)\right| \ , \tag{3.33}$$

It will thus be observed that varying the phase, β, is a technique which will allow the direction of the array factor to be altered – i.e., β may be altered to "steer" the beam of the array.

Special Case 1. $\beta = 0°$ for the Two-Element Array

Let us suppose that the equal magnitude currents that led to (3.33) are also in phase. That is, $\beta = 0°$. Furthermore let us assume that the array spacing is $d = \lambda/2$ so that $kd = \pi$.

Horizontal Plane ($\alpha = \phi$):

We note from (3.33) that

$$|E'(\alpha,\beta)|_{\substack{\alpha=\phi \\ \beta=0°}} = 2\left|\cos\left(\frac{kd\cos\alpha+\beta}{2}\right)\right|_{\substack{\alpha=\phi \\ \beta=0°}} = 2\left|\cos\left(\frac{\pi\cos\phi}{2}\right)\right|$$

Note:

Note that the maxima are in the direction perpendicular to the array line (which was taken along the x-axis from the outset of this analysis). Thus, with $\beta = 0$, that is with the currents in phase, the array is said to operate in *broadside* mode.

Vertical Plane ($\alpha = 90° - \theta$):

Special Case 2. $\beta = 180°$ for the Two-Element Array

Let us suppose that the equal magnitude currents that led to (3.33) are out of phase by 180° – that is, $\beta = 180°$. Furthermore let us assume that the array spacing is still $d = \lambda/2$ so that $kd = \pi$.

Horizontal Plane ($\alpha = \phi$):

We note from (3.33) that the group pattern is given by

$$|E'(\alpha,\beta)|_{\substack{\alpha=\phi \\ \beta=180°}} = 2\left|\cos\left(\frac{kd\cos\alpha+\beta}{2}\right)\right|_{\substack{\alpha=\phi \\ \beta=180°}} = 2\left|\cos\left(\frac{\pi\cos\phi+\pi}{2}\right)\right|$$

Comparing with the $\beta = 0°$ case, we note that the maxima are in the direction parallel to the array line (which was taken along the x-axis from the outset of this analysis). Thus, with $\beta = 180°$, that is with the currents out of phase by 180°, the array is said to operate in *endfire* mode.

Vertical Plane ($\alpha = 90° - \theta$):

Now the group pattern becomes

$$|E'(\alpha,\beta)|_{\substack{\alpha=90°-\theta \\ \beta=180°}} = 2\left|\cos\left(\frac{\pi\sin\theta+\pi}{2}\right)\right|$$

It is possible to determine that when $d > \lambda/2$, additional lobes appear. (Try this with a small Matlab program).

3.3.2 The Linear Array

Consider, next, the more general case of M equally-spaced identical elements lying along a line as indicated. Again, we are interested in the far field. As usual, d is the element spacing, r_0 is the distance from the 0^{th} element and so on up to r_{M-1}.

Assumption: The array length $(M-1)d \ll r_n$ for $0 \le n \le M-1$.

Consequence: The θ and ϕ coordinates of the field point are the same with respect to each element.

Thus, the total field at the observation point due to all of the elements may be written as

$$\vec{E}_T = \sum_{n=0}^{M-1} \vec{E}_n(r_n, \theta, \phi) \tag{3.34}$$

Proceeding as for the two-element case (see equation (3.26))

$$\tag{3.35}$$

and, in general, for the n^{th}

$$\begin{aligned} \vec{E}_n(r_n, \theta, \phi) &= \\ &= \end{aligned} \tag{3.36}$$

since the direction-dependent factor is the same for each element. As before, in the amplitude sense, for the far field,

$$\frac{1}{r_n} \approx \frac{1}{r_0}\,, \tag{3.37}$$

but in the phase term we must specify

$$r_n = r_0 - nd\cos\alpha\ . \tag{3.38}$$

Using (3.36), (3.37), and (3.38) in (3.34),

$$\begin{aligned}\vec{E}_T &= \sum_{n=0}^{M-1} \frac{CI_n}{r_0} e^{-jkr_0 + jknd\cos\alpha} \vec{E}_{0a}(\theta,\phi) \\ &= \left[\frac{CI_0 e^{-jkr_0}}{r_0} \vec{E}_{0a}(\theta,\phi)\right]\left[\sum_{n=0}^{M-1} \frac{I_n}{I_0} e^{jknd\cos\alpha}\right]\end{aligned}$$

Therefore,

$$\vec{E}_T = \vec{E}_0(r_0,\theta,\phi)\left[\sum_{n=0}^{M-1} \frac{I_n}{I_0} e^{jknd\cos\alpha}\right] \tag{3.39}$$

The term containing the summation is commonly referred to as the *array factor* (AF). It may be clearly observed from equation (3.39) that the following statement may be made about the "resultant pattern" of a linear array:

$$\boxed{\text{resultant pattern} = \text{ unit pattern} \times \text{ group pattern.}}$$

Now, some special cases of linear arrays may be considered.

Special Case 1: The Uniform Linear Array

In a *uniform* linear array, the current magnitudes are equal, but there is a uniform progression in the phase of the current from one element to the next. This phase shift, β, may take positive or negative values. For this case, the currents may be written as

$$I_n = I_0 e^{jn\beta}\ , \tag{3.40}$$

Then, equation (3.39) may be written as

$$\vec{E}_T = \vec{E}_0(r_0,\theta,\phi)E'(\alpha,\beta)$$

where

$$E'(\alpha,\beta) = \sum_{n=0}^{M-1} e^{jn(\beta + kd\cos\alpha)}$$

or

$$E'(\alpha,\beta) = \sum_{n=0}^{M-1} e^{jn\Psi} \tag{3.41}$$

where

$$\Psi = \beta + kd\cos\alpha \ . \tag{3.42}$$

The RHS of (3.41) is a M-term geometric progression whose first term is 1 and whose constant ratio is $e^{j\Psi}$. Therefore, (3.41) may be written as

Since $\left|e^{j\frac{\Psi}{2}(M-1)}\right| = 1$, it is seen that

$$|\mathrm{AF}| = |E'(\alpha,\beta)| = \left|\frac{\sin\left(\frac{M\Psi}{2}\right)}{\sin\left(\frac{\Psi}{2}\right)}\right| \tag{3.43}$$

will establish the shape of the group pattern. Furthermore, if the unit pattern is circular – as is the case, for example, for $\lambda/2$ vertical dipoles with $\alpha = \phi$ and $\theta = 90°$ – $|\mathrm{AF}|$ establishes the resultant pattern for the array.

Equation (3.43) is worthy of closer scrutiny:

1. Notice that when $\Psi = 0$, (3.43) becomes the indeterminate form $\frac{0}{0}$. However, on using L'Hopital's rule, it readily verified that

$$\lim_{\Psi\to 0} |\mathrm{AF}| = M.$$

   It may indeed be verified that $\Psi = 0$ produces the *principal* (or absolute) *maximum* (check it out numerically). Thus, a normalized form of (3.43) having a principal maximum of unity may be written as

$$|\mathrm{AF}|_n = \frac{1}{M}\left|\frac{\sin\left(\frac{M\Psi}{2}\right)}{\sin\left(\frac{\Psi}{2}\right)}\right| \tag{3.44}$$

2. *Secondary maxima* occur when, in (3.43),

$$M\Psi = \pm(2i+1)\pi \ , \quad i \in \mathrm{N}$$

or

Therefore from (3.42)

which gives

$$\boxed{\cos\alpha = -\left(\frac{\beta}{kd}\right) \pm (2i+1)\frac{\pi}{Mkd}} \tag{3.45}$$

$i = 1$ produces the first secondary maximum, $i = 2$ produces the second secondary maximum, and so on. It is easy to show that for large M, the first secondary maximum is about -13.5 dB down from the principal maximum. (To be completed on a tutorial).

3. Nulls occur when, in (3.43), the numerator is zero but the denominator is not zero. With this caution, we set

$$\frac{M\Psi}{2} = \pm i\pi \ , \quad i \in \mathrm{N} \ \ (\text{but } i \neq M, 2M, 3M, \ldots)$$

which implies

or

$$\boxed{\cos\alpha = -\left(\frac{\beta}{kd}\right) \pm \left(\frac{2i\pi}{Mkd}\right)} \tag{3.46}$$

Remember that in (3.45) and (3.46), $\alpha = \phi$ in the horizontal plane and $\alpha = 90° - \theta$ in the vertical plane. This analysis shows that, in general, a linear array will produce an array factor with a shape something like

Case 1a. The Uniform Broadside Array

Consider M vertical equally spaced dipoles along the x-axis as shown. We are going to examine the array factor in the x-y plane – i.e. $\alpha = \phi$.

Recall that the normalized unit pattern is a unit circle in the x-y plane. In a *broadside* linear array, the principal maximum is in a direction 90° to the line containing the array elements – that is, $\phi = 90°$ is the direction of the *principal maximum.* Let's check the required phase for the currents:

We know from (3.42) that

$$\Psi = \beta + kd\cos\alpha$$

and note in passing that Ψ is an even function of ϕ for the horizontal plane, making the AF an even function also. For $\alpha = \phi$ and $\phi = \pi/2$ along with the principal maximum occurring at $\Psi = 0$ (see Note (1) above) we have

Since k and d are constants, $\beta = 0$. Therefore, when the elements are fed "in-phase", the array operates in "broadside" mode.

Typical Result:

Width of the Main Lobe – i.e. Beamwidth

Let's consider the width of the main lobe in terms of the E-field pattern. With reference to the sketch above,

$$\text{Width of Main Lobe } = 2\Delta\phi$$

where $\Delta\phi$ is the angle between the principal maximum and the first null either side of it. The direction of the first null is readily obtained from (3.46) as

Therefore,

$$\text{Width of Main Lobe } = 2\sin^{-1}\left[\frac{\lambda}{Md}\right] \tag{3.47}$$

We make the important observation from (3.47) that if the array is long (i.e. Md is large), the "beam" will be narrow. In fact, if we assume (λ/Md) is small so that

$$\sin^{-1}\left[\frac{\lambda}{Md}\right] \approx \frac{\lambda}{Md} ,$$

then

$$\text{Width of Main Lobe } = 2\Delta\phi = \frac{2\lambda}{Md}.$$

Half-Power Beam Width

We now attempt to find the half-power beamwidth of a uniform broadside linear array of dipoles for which, in the horizontal plane as considered above, the unit pattern is a circle. Recall that for this situation $\beta = 0$ and $\alpha = \phi$ so that (3.42) is

$$\Psi = kd\cos\phi$$

and that the E-field pattern has array factor with maximum value of M. Recalling that

$$\text{power } \propto (E\text{-field})^2,$$

a normalized power pattern may be found by simply squaring the array factor:

For the situation under consideration,

$$|\mathrm{AF}|^2 = \left| \frac{\sin\left(\frac{Mkd}{2}\cos\phi\right)}{\sin\left(\frac{kd}{2}\cos\phi\right)} \right|^2$$

whose maximum value is obviously M^2. To determine where this quantity reaches half its maximum value (i.e. the direction in which the power is half its maximum), we set

$$\tag{3.48}$$

and "solve" for ϕ eventually. Here we have used $\Psi_{1/2}$ to indicate the Ψ at the half-power positions. It is not hard to convince oneself that for a long array (say, several elements spaced by half wavelengths) $\Psi_{1/2}/2$ is small so that $\sin(\Psi_{1/2}/2) \approx \Psi_{1/2}/2$. Also, recall that

and the numerator in the left member of equation (3.48) gives (using the first two terms of the expansion)

$$\sin\left(\frac{M\Psi_{1/2}}{2}\right) = \frac{M\Psi_{1/2}}{2} - \frac{1}{6}M^3\left(\frac{\Psi_{1/2}}{2}\right)^3 \tag{3.49}$$

Rearranging (3.48)

and using the above approximations gives

This reduces to

$$\Psi_{1/2} = \frac{2.65}{M}$$

Since $\beta = 0$, using (3.42) with $\alpha = \phi$ gives

$$\tag{3.50}$$

where $\phi_{1/2}$ is the angle associated with the half-power point. (Note that in (3.50), (Md) is approximately the array length.) To see how this analysis leads to the half-power beamwidth, consider the following:

From this we see that for large array lengths, $\Delta\phi_{1/2}$ will be small. To a good approximation, therefore,

and the half-power beamwidth, $\mathrm{BW}_{1/2}$, is

$$\boxed{\mathrm{BW}_{1/2} = 2\Delta\phi_{1/2} = \frac{2.65\lambda}{\pi M d}} \tag{3.51}$$

Notice that the half-power beam width is inversely proportional to the array length (approximately).