Unit 2

Small Antennas and Some Antenna Parameters

2.1 The Fundamental Source of Radiated Energy

We have seen from Maxwell's equations that a time-varying $\vec{B}$ field (or $\vec{H}$ field) implies a time-varying $\vec{E}$ field (or $\vec{D}$ field) and vice versa. Earlier it would have been observed that one solution to these equations represented (plane) wave energy propagating through some space (free or otherwise). We know that steady (dc) currents produce steady magnetic fields. It should be obvious, then, that time-varying currents will produce time-varying $\vec{H}$ fields and, thus, also time-varying $\vec{E}$ fields. Clearly, a time-varying current necessitates an *accelerating charge*. Therefore, to have e-m energy radiating away from a source, the source must consist of accelerating charge(s) – a stationary charge or one moving with a constant velocity does not radiate. There are several simple ways in which a charge may accelerate and thus radiate:

1. charge may oscillate with simple harmonic motion along a straight wire;

2. charge may move with a constant speed along a bent or curved wire;

3. charge may encounter a termination (load), a truncation (wire not infinite), or a discontinuity (eg., going from one wire to another of different size or electrical parameters). In these cases, electric charge may be caused to change direction –i.e., reflect. Many such possibilities exist – wanted and unwanted.

Illustration: See Figure 1.10 of the Balanis text.

To quantify this idea, consider a packet of electric charge moving along a straight thin conductor in the z-direction, say.

Let $\rho_L \equiv$ linear charge density in C/m

and $v_z \equiv$ velocity in the $\hat{z}$ direction.

Then the current, I, is given by

$$I = \frac{dQ}{dt}$$

where Q is charge and t is time. Since $dQ = d(\rho_L z)$, and if ρ_L is constant in time so that $dQ = \rho_L dz$, the current may be written as

$$I = \frac{\rho_L dz}{dt} = \rho_L v_z \ .$$

Taking the derivative w.r.t. time,

$$\frac{dI}{dt} = \rho_L \frac{dv_z}{dt} = \rho_L a_z \ .$$

where a_z is acceleration. For a pulse of length ℓ, we may multiply this result by ℓ to get

$$\ell \frac{dI}{dt} = \ell \rho_L a_z$$

or using "overdot" for the time derivative

$$\boxed{\ell \dot{I} = Q \dot{v}_z}$$

$Q = \ell \rho_L$ is the charge in the pulse of length ℓ. For radiation to occur, this equation must hold – i.e., current must have a non-zero time derivative ($\dot{I} \neq 0$ or $\ddot{Q} \neq 0$)– charges must accelerate.

2.2 The infinitesimal Dipole

2.2.1 Reality and Useful "Fiction"

In transmission line theory, it may be noted that a two-wire open-circuit transmisision line could be configured as a "dipole" antenna.

Illustration:

In this situation (i.e., for $\ell = \lambda/2$), the current, I, has a node at each end of the wire, the amplitude decreasing continually away from the centre. The wavelenth of I is λ. For other lengths, the current pattern may be more complicated; however, current nulls or nodes still occur at the wire ends for the infinitesimally thin dipole.

There exist methods of making the current more uniform. Consider, first, a $\ell = \lambda/4$ *monopole* antenna which is fed against an ideal ground plane as shown:

As we'll discuss in more detail later, there is a $\lambda/4$ reflection in the perfect ground. This monopole will resonate at the same frequency and provide the same pattern shape as a $\lambda/2$ dipole (more later on this). Notice that there is still a current null at the top of the antenna. The shape of the current pattern may be modified electrically by shortening the antenna somewhat and base-loading it with an inductance to provide the same resonance. Across the inductor section (see illustration below), the voltage is essentially linear and the current uniform, making for a more uniform overall current loop.

Another method of forcing the current to be more uniform across the antenna is by end-loading (or top-loading) it with a capacitance, for example a capacitive disc.

With the idea of a *uniform current distribution* in mind, we are next going to consider an infinitesimal dipole ($\ell \ll \lambda$) carrying such a distribution and oriented as shown:

Letting I_0 be the current magnitude (and it *may be* complex),

$$\vec{I}(z') = I_0 \hat{z} \tag{2.1}$$

(You may visualize how such a structure might be used in a repeated way to give an extended antenna element).

I "constant" over a differential element but may vary from one element to the next over the length of the wire.

Thus, this infinitesimal dipole idea, while ficticious in nature, will be very useful in "building up to" a real situation.

2.2.2 The Vector Potential for a Uniform-Current Element

Consider again the infinitesimal, uniform-current, dipole – sometimes referred to as a Hertzian dipole – oriented as previously:

Note that $x' = 0$ and $y' = 0$. We wish to determine the vector potential with a view to obtaining the $\vec{E}$ and $\vec{H}$ fields. In equation (1.40) for the vector potential, we replace $\vec{J}(\vec{r}')dv'$ with

$$I(z')dz' \, \hat{z}$$

since the "source" is oriented along the z-axis and located as shown. Thus, the volume integral reduces to a line integral and

$$\tag{2.2}$$

Using $\vec{I}(z')$ from equation (2.1),

$$\vec{A}(\vec{r}) = \hat{z}\frac{\mu I_0}{4\pi}\int_{-\ell/2}^{\ell/2}\frac{e^{-jk\sqrt{x^2+y^2+(z-z')^2}}}{\sqrt{x^2+y^2+(z-z')^2}}dz' \tag{2.3}$$

While the integration in (2.3) is not available in closed form (and thus neither are the $\vec{E}$ and $\vec{H}$ fields), it does yield to some useful approximations. CAUTION: *In antenna theory, very little of the often formidable analysis can be carried out exactly – i.e., in closed form.*

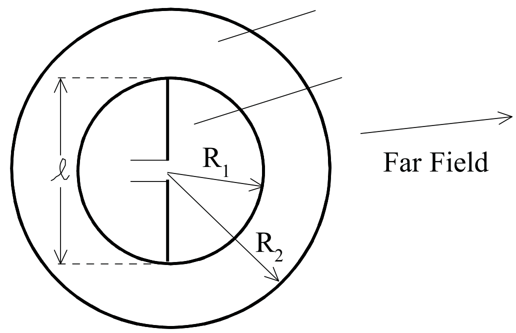
Before considering (2.3) further, we note the following terms associated with fields radiated by antennas:

1. "The Reactive" Near-Field Region is that portion of the near-field region immediately surrounding the antenna wherein the reactive field predominates. The energy in this field does not radiate away from the antenna but rather oscillates near the antenna like in a resonator. It is "trapped" near the antenna.

2. The Radiating Near-Field (or Fresnel Zone) is that region of the antenna field between the reactive near-field and the far-field where radiation (rather than reactive) fields are predominate and wherein the angular field distribution is dependent on the distance from the antenna. If the antenna's maximum dimension is much smaller than a wavelength, this region may not exist.

3. The Far-Field (or Fraunhofer) Region is that region of the radiation antenna field where the field distribution is essentially independent of the distance from the antenna. This is the "zone" in which we will be interested in this course.

The distances "broadly" dividing these regions are depicted below for $\ell > \lambda$:



Approximation of $\vec{A}(\vec{r})$:

Now we return to the vector potential of equation (2.3) and seek to determine its form for the Hertzian dipole (infinitesimal dipole). This will lead us to the $\vec{E}$ and $\vec{H}$ fields for such a dipole. Consider first the square root appearing in the integrand:

Therefore for the Hertzian dipole,

$$\sqrt{x^2 + y^2 + (z - z')^2} \approx r - z' \cos\theta \; . \tag{2.4}$$

Using (2.4) in (2.3) gives

$$\vec{A}(\vec{r}) \approx \hat{z}\frac{\mu I_0}{4\pi}\int_{-\ell/2}^{\ell/2}\frac{e^{-jk(r-z'\cos\theta)}}{r-z'\cos\theta}dz' \ .$$

Again, because $r \gg z'$, in the amplitude portion of the integrand, we may use $r - z'\cos\theta \approx r$ to get

$$\vec{A}(\vec{r}) \approx \hat{z}\frac{\mu I_0}{4\pi}\int_{-\ell/2}^{\ell/2}\frac{e^{-jk(r-z'\cos\theta)}}{r}dz' \ .$$

Notice that the last approximation has not been used in the phase expression – WHY NOT?

At a particular position dictated by $\vec{r}$, we have

$$\boxed{\vec{A}(\vec{r}) \approx \hat{z}\left(\frac{\mu I_0 \ell}{4\pi r}\right)e^{-jkr}\,\mathrm{Sa}\left[\left(\frac{k\ell}{2}\right)\cos\theta\right]} \tag{2.5}$$

Here Sa is the familiar sampling function $\left(\mathrm{Sa}(x) = \dfrac{\sin x}{x}\right)$.

<u>Illustration:</u>

Since we are dealing with the infinitesimal dipole it seems reasonable to further consider the situation

$$\left| \frac{kl}{2} \cos\theta \right| \ll 1$$

or even

$$\frac{kl}{2} \ll 1 \quad \Rightarrow$$

$$\Rightarrow$$

$$\Rightarrow$$

From the note on the sampling function, in this case,

$$\mathrm{Sa}\left(\frac{kl}{2} \cos\theta \right) \approx 1 \; . \tag{2.6}$$

To use the approximation to the infinitesimal dipole, a rule of thumb stipulates that the overall antenna length must be very small – usually, $\ell \leq 0.02\lambda$. Also, remember that for this analysis $I(z')$ is a constant.

Using approximation (2.6), equation (2.5) becomes for the vector potential of the infinitesimal dipole

$$\boxed{\vec{A}(\vec{r}) \approx \hat{z} \left(\frac{\mu I_0 \ell}{4\pi r} \right) e^{-jkr}} \tag{2.7}$$

The governing assumptions are that $\ell \ll \lambda$ and $\ell \ll r$, the last assumption being ensured by our stipulation that $r \gg z'$. The form of equation (2.7) suggests a spherical field – a "wave" travelling radially outward if we assume a $e^{j\omega t}$ time dependency. Again, it must be emphasized that this is the infinitesimal dipole result.

2.2.3 The Electric and Magnetic Fields and Far-Fields for the Infinitesimal Dipole

It is easily deduced from Section 1.3 (Do This) that

$$\tag{2.8}$$

and we have also, by definition, that $\vec{B} = \vec{\nabla} \times \vec{A}$ or

$$\vec{H} = \frac{1}{\mu}\vec{\nabla} \times \vec{A}\,. \tag{2.9}$$

As an exercise, $\vec{E}$ and $\vec{H}$ will be found in spherical coordinates (using (2.7)–(2.9) and the appropriate coordinate transformations) to be given by

$$E_r = \eta\frac{I_0\ell\cos\theta}{2\pi r^2}\left[1 + \frac{1}{jkr}\right]e^{-jkr} \tag{2.10}$$

$$E_\theta = j\eta\frac{kI_0\ell\sin\theta}{4\pi r}\left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right]e^{-jkr} \tag{2.11}$$

$$E_\phi = 0 \tag{2.12}$$

$$H_r = H_\theta = 0 \tag{2.13}$$

$$H_\phi = j\frac{kI_0\ell\sin\theta}{4\pi r}\left[1 + \frac{1}{jkr}\right]e^{-jkr} \tag{2.14}$$

<u>The Far-Field $\vec{E}$:</u>

Consider, first, E_θ. As long as k is non-negligible, we may stipulate the far field by $r \gg 1$. Then the $1/r^2$ and $1/r^3$ terms in (2.11) will be negligible in comparison to the $1/r$ term. Therefore, in the far-field

$$E_\theta \approx \frac{j\eta kI_0\ell\sin\theta}{4\pi r}e^{-jkr}\,. \tag{2.15}$$

Furthermore, $E_r \ll E_\theta$ since the former contains only $1/r^2$ and $1/r^3$ terms. Since $E_\phi = 0$, we see that the E_θ field dominates the $\vec{E}$-field radiation field of the infinitesimal dipole. To emphasize this, we write that

$$\vec{E} = E_r\hat{r} + E_\theta\hat{\theta} + E_\phi\hat{\phi}$$

to a good approximation becomes

$$\vec{E} \approx E_\theta\hat{\theta}$$

or, for future reference

$$\boxed{\vec{E} \approx \left[\frac{j\eta kI_0\ell e^{-jkr}}{4\pi r}\right]\sin\theta\,\hat{\theta}} \tag{2.16}$$

The Far-Field $\vec{H}$: From equations (2.13) and (2.14),

$$\vec{H} = H_r\hat{r} + H_\theta\hat{\theta} + H_\phi\hat{\phi} = H_\phi\hat{\phi}.$$

Using $r \gg 1$ we have

$$\boxed{\vec{H} \approx \left[\frac{jkI_0\ell e^{-jkr}}{4\pi r}\right]\sin\theta\,\hat{\phi}} \tag{2.17}$$

Observations:

1. Both $\vec{E}$ and $\vec{H}$ have a $\sin\theta$ dependency, indicating the presence of maxima and minima in the field of observation (no spherical symmetry as for the vector potential A).

2. There is no ϕ dependency in either $\vec{E}$ or $\vec{H}$.

3. $\vec{E}\times\vec{H}$ is in the $\hat{\theta}\times\hat{\phi} = \hat{r}$ direction, indicating a wave whose $\vec{E}$ and $\vec{H}$ are in the direction of propagation and are mutually perpendicular (recall the Poynting vector). We note that this is referred to as a transverse electromagnetic wave (TEM wave).

4. From observations (1) and (2) we see that the far-field $\vec{E}$ and $\vec{H}$ space patterns in the "vertical" plane are *doughnut-shaped*.

5. The space containing the dipole is unbounded and homogeneous. The intrinsic impedance of the space from (2.16) and (2.17) is given by

$$\eta = \frac{E_\theta}{H_\phi}\ .$$

If we have the current element in free space, of course, $\mu = \mu_0$ and $\epsilon = \epsilon_0$ and $\eta = \eta_0 \approx 120\pi$.

6. It may be argued from (2.16) or (2.17) that at a particular instant in time an observer at $\vec{r}$ "sees" what was happening at the antenna r/c seconds earlier. Consider, for example, (2.17). Recalling that $k = \omega/c$ (assuming free space), equation (2.17) may be cast as

$$\vec{H} \approx \frac{j\omega}{c}\left[\frac{I_0 \ell e^{-j\omega\frac{r}{c}}}{4\pi r}\right]\sin\theta\,\hat{\phi}$$

We note

(a) $j\omega/c$ transforms to $(1/c)\partial/\partial t$ in the time domain;

(b) If

$$\mathcal{F}(\omega) \Leftrightarrow f(t)$$

is a Fourier transform pair, then

$$\mathcal{F}(\omega)e^{-j\omega a} \Leftrightarrow f(t-a)$$

where, here, $a = r/c$.

Thus, noting that I_0 is a phasor in the above expression for $\vec{H}$ (which is really $\vec{H}(\vec{r},\omega)$) transforms in the temporal sense to

$$\vec{H}(\vec{r},t) \approx \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{i\left(t-\frac{r}{c}\right)\ell}{4\pi r}\right]\sin\theta\,\hat{\phi}$$

where $i(t)$ is the time domain current. As expected, the $\vec{H}$ field observed at $\vec{r}$ at time t depends on the value of the current at $(t - \frac{r}{c})$.

It must be emphasized that the results of the preceeding analysis are useful for very short (in terms of wavelength) antennas provided that the current is uniform.

2.2.4 Power Density and Radiated Power

Power Density:

We now consider the nature of the power associated with the radiation fields calculated in (2.16) and (2.17). Initially, we note that in the time domain

$\vec{E}$ and $\vec{H}$ being the phasor fields given by (2.16) and (2.17). Recall that the Poynting vector, $\vec{\underset{\sim}{\mathcal{P}}}$, is given by

$$\vec{\underset{\sim}{\mathcal{P}}} = \vec{\underset{\sim}{E}} \times \vec{\underset{\sim}{H}} \tag{2.18}$$

and is the power per unit area (i.e., *power density*) directed perpendicular to both $\vec{\underset{\sim}{E}}$ and $\vec{\underset{\sim}{H}}$. That is, $\vec{\underset{\sim}{\mathcal{P}}}$ is in the direction of propagation for the far field region.

$$\vec{\underset{\sim}{\mathcal{P}}} = \frac{1}{2}\mathcal{R}e\left\{\vec{E} \times \vec{H}^*\right\} + \frac{1}{2}\mathcal{R}e\left\{\vec{E} \times \vec{H}e^{j2\omega t}\right\} \tag{2.19}$$

Here, $\vec{E}$ and $\vec{H}$ are themselves phasors – not time dependent. Thus the first term in (2.19) is time-independent, while the second term is time-harmonic. Since the *time-average* of a time-harmonic quantity is zero, the time-averaged Poynting vector is given simply as

$$\vec{\mathcal{P}}_a = \frac{1}{2}\mathcal{R}e\left\{\vec{E} \times \vec{H}^*\right\} \ . \tag{2.20}$$

The second term is simply an oscillatory (reactive) power density with radian frequency 2ω. We note in passing that if the complete expressions given by (2.11) and (2.14) were used for the fields, the time-averaged *radiated* power density would still be given by (2.20).

Carrying out (2.20) using (2.16) and (2.17) we have

This is the time-averaged (real) radiated power density of the current element. Note that $|I_0|^2 = I_0 I_0^*$ for a complex current I_0. In general then

$$\vec{\mathcal{P}}_a = \frac{\eta k^2 |I_0|^2 \ell^2}{32\pi^2 r^2} \sin^2\theta \hat{r} = \frac{1}{2\eta} |\vec{E}|^2 \hat{r} \ . \tag{2.21}$$

This is the *radiation power density* in any $\hat{r}$ direction. Therefore, the 3-d *shape* of $\vec{\mathcal{P}}_a$ is "built up" by rotating $\sin^2\theta$ in the z-y plane (for example – or any other constant ϕ plane) about the z-axis.

Illustration:

Power Radiated (P_r):

Consider a spherical "surface" of radius r enclosing the current element:

Vector differential surface area is:

$d\vec{S}_r = dS_r \, \hat{r} = r^2 \sin\theta d\theta d\phi \, \hat{r}$

Also, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

Since $|\vec{\mathcal{P}}_a|$ in (2.21) is the power per unit area, P_r is obtained by a simple integration of $\vec{\mathcal{P}}_a$ over the spherical surface.

$$P_r =$$

$$=$$

Thus, since the remaining integral has a value of 4/3,

$$P_r = \frac{\eta k^2 |I_0|^2 \ell^2}{12\pi} \tag{2.22}$$

in watts. This analysis is fine for "short" antennas (i.e. short compared to a wavelength) with *uniform* current distribution. (A rule of thumb is $\ell \leq \lambda/50$).

2.2.5 Near and Intermediate Fields of the Hertzian Dipole

Under the assumptions of a Hertzian Dipole ($\ell << \lambda$, radius $<< \lambda$) we have shown that the fields may be approximated as

$$E_r = \eta \frac{I_0 \ell \cos\theta}{2\pi r^2} \left[1 + \frac{1}{jkr}\right] e^{-jkr} \tag{2.23}$$

$$E_\theta = j\eta \frac{k I_0 \ell \sin\theta}{4\pi r} \left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right] e^{-jkr} \tag{2.24}$$

$$E_\phi = 0 \tag{2.25}$$

$$H_r = H_\theta = 0 \tag{2.26}$$

$$H_\phi = j \frac{k I_0 \ell \sin\theta}{4\pi r} \left[1 + \frac{1}{jkr}\right] e^{-jkr} \tag{2.27}$$

It is customary to define the "nature" of the fields in terms of *radian distance* $r = \lambda/2\pi$ from the source. Clearly, the radian distance implies

$$kr = 1 \ .$$

The *near field* is defined as that region for which $kr << 1$, while the *intermediate field* requires $kr > 1$ and in the far field $kr >> 1$. Read Section 4.2.3 of the text, pp 156-157.

2.2.5.1 Near-Field ($kr << 1$)

Examining equation (2.23), it is clear that for the near-field condition, the dominant term in E_r is that involving $\frac{1}{jkr}$. Thus, we approximate

$$E_r \approx -j\frac{\eta I_0 \ell e^{-jkr}}{2\pi k r^3}\cos\theta \tag{2.28}$$

Similarly, the remaining fields may be approximated as

$$E_\theta \approx -j\frac{\eta I_0 \ell e^{-jkr}}{4\pi k r^3}\sin\theta \tag{2.29}$$

$$E_\phi = 0 \;\; ; \;\; H_r = H_\theta = 0 \tag{2.30}$$

$$H_\phi \approx \frac{I_0 \ell e^{-jkr}}{4\pi r^2}\sin\theta \tag{2.31}$$

It may be immediately observed that the E-field and H-field are in "phase quadrature" and thus no time-average power will result for this near field. This is easily verified by showing (DO THIS)

$$\vec{\mathcal{P}}_a = \frac{1}{2}\mathcal{R}e\left\{\vec{E}\times\vec{H}^*\right\} = 0 \tag{2.32}$$

2.2.5.2 Intermediate-Field ($kr > 1$)

Notice that this does not say that $kr >> 1$. An examination of equation (2.23) reveals the first term in square brackets to be dominant for $kr > 1$ and

$$E_r \approx \frac{\eta I_0 \ell e^{-jkr}}{2\pi r^2}\cos\theta \tag{2.33}$$

Similarly, the remaining fields may be approximated as

$$E_\theta \approx \frac{j\eta k I_0 \ell e^{-jkr}}{4\pi r}\sin\theta \tag{2.34}$$

$$E_\phi = 0 \;\; ; \;\; H_r = H_\theta = 0 \tag{2.35}$$

$$H_\phi \approx j\frac{k I_0 \ell e^{-jkr}}{4\pi r}\sin\theta \tag{2.36}$$

From equations (2.34) and (2.36), we observe that there is now a time-average radial power flow, but from (2.33) and (2.36) a "reactive" component is also evident. This is easily established in the usual way:

$$\begin{aligned}\vec{\mathcal{P}}_a &= \frac{1}{2}\mathcal{R}e\left\{\vec{E}\times\vec{H}^*\right\}\\ &= \frac{1}{2}\mathcal{R}e\left\{\left(E_r\,\hat{r}+E_\theta\,\hat{\theta}\right)\times H_\phi^*\hat{\phi}\right\}\\ \vec{\mathcal{P}}_a &= \frac{1}{2}\mathcal{R}e\left\{-E_rH_\phi^*\hat{\theta}+E_\theta H_\phi^*\hat{r}\right\}\ . \qquad (2.37)\end{aligned}$$

Noting that

$$E_rH_\phi^* = -\frac{j\eta k|I_0|^2\ell^2}{8\pi^2r^3}\sin\theta\cos\theta$$

and that

$$E_\theta H_\phi^* = \frac{\eta k^2|I_0|^2\ell^2}{16\pi^2r^2}\sin^2\theta\ ,$$

it is obvious that for the intermediate field

$$\vec{\mathcal{P}}_a = \frac{1}{2}E_\theta H_\phi^*\hat{r} \qquad (2.38)$$

while the $\hat{\theta}$ component is "reactive" (is it "inductive" or "capacitive" or a combination?).

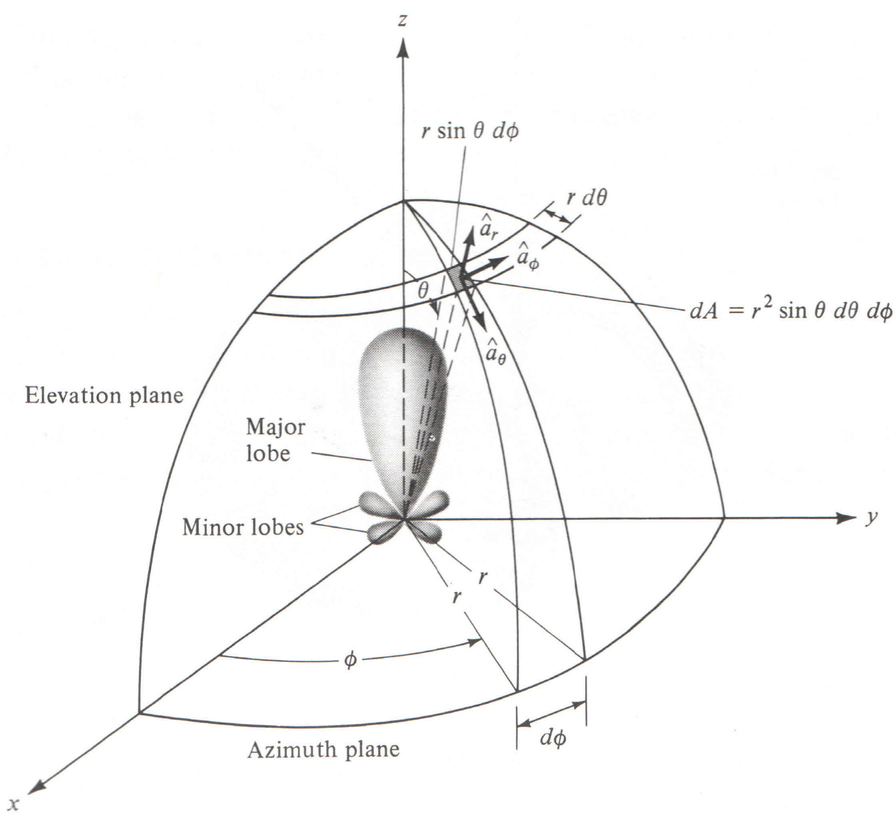
Exercise: Show that the E-field vector for the intermediate field traces out an ellipse (in time), the plane of rotation being parallel to the propagation direction. (This is referred to as the *cross field*.)

2.3 General Antenna Parameters and Characteristics

Having examined the far-field characteristics of an infinitesimal dipole, we now consider some (in no way an exhaustive list) features common to all antennas. For now, whenever possible, we'll relate them to the foregoing small-current-element analysis. In fact, we have seen some of the following already. The definitions may be found in the IEEE Standard Definitions Terms for Antennas, *IEEE Trans. on Ant. and Prop., vol. AP-31, No. 6, Part II*, Nov., 1983.

2.3.1 Radiation Pattern and Related Concepts

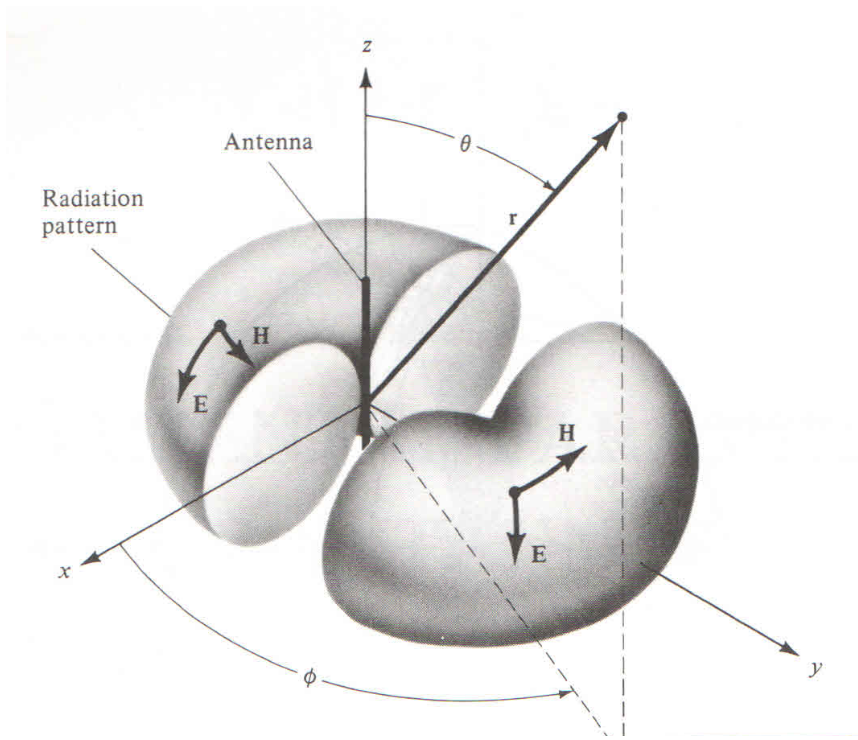
An *antenna radiation pattern* or simply *antenna pattern* is defined as "a mathematical function or graphical representation of the radiation properties of the antenna as a function of the space coordinates". In most cases, this pattern is determined in the far-field context – this is exclusively the case in this course. The term "radiation properties" includes power flux density, radiation intensity, field strength, directivity, phase and polarization. We will consider some of these. A suitable coordinate system with some terminology is shown below (taken from Balanis, Fig. 2.1).



1. Power and Field Patterns, Polarization and Principal Plane Patterns

(i.) A trace of the received power at a constant radius from the antenna is called the *power pattern.* A similar trace of the magnetic (or electric) field along a constant radius is called an *amplitude field pattern.* In Sections 2-2-3 and 2-2-4, we alluded to these for the infinitesimal dipole. The power patterns for this and more general structures are shown below (see Balanis, Figures 2.6 and 2.3).

Pattern for infinitesimal dipole:



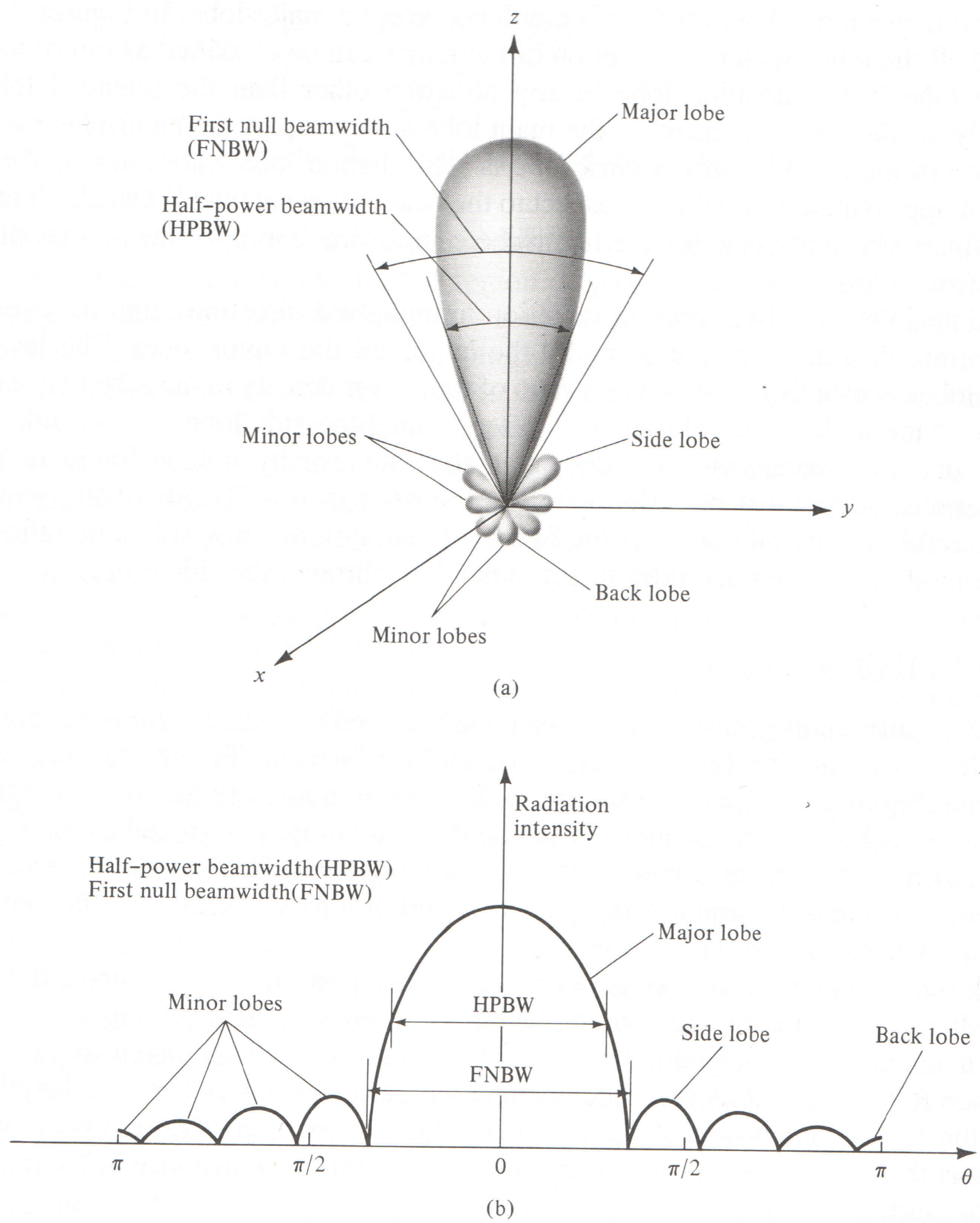
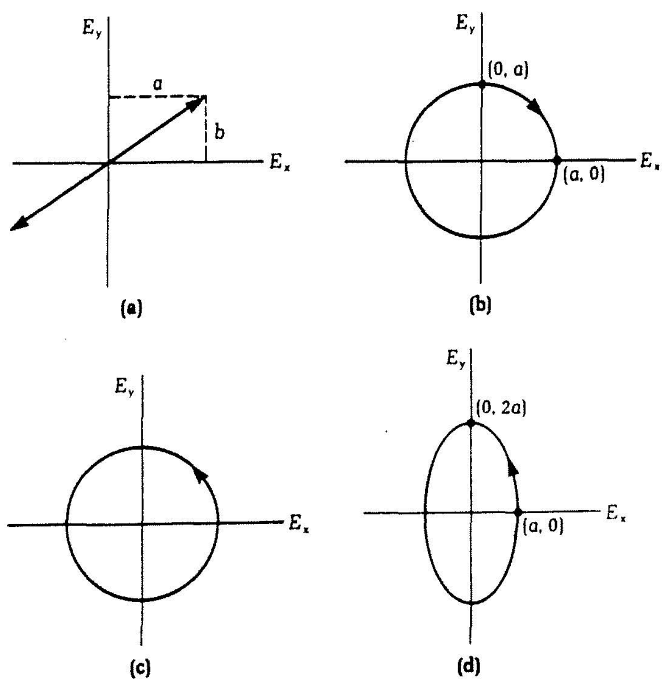


Figure 2.4 (a) Radiation lobes and beamwidths of an antenna pattern. (b) Linear plot of power pattern and its associated lobes and beamwidths.

(ii.) Polarization: There are several aspects to the notion of polarization in e-m theory.

First, we define the *polarization of a radiated wave* as that property of an e-m wave describing the time-varying direction and relative magnitude of the electric field vector. Specifically, *at a fixed postion in space*, the locus of the tip of the $\underset{\sim}{\vec{E}}$-field vector and the sense in which this figure is traced as observed along the direction of propagation define the polarization of the wave. The general polarization is *elliptical*, which degenerates in important special cases to *linear* or *circular* polarization. A simplified sketch (taken from Applied Electromagnetism, 3$^{\text{rd}}$ edition, by L.C. Shen and J.A. Kong, PWS Publishing Company, Boston, 1995) is shown below. The elliptical or circular polarizations may be left-handed or right-handed, depending on the hand for which the curl of the fingers is in the direction of rotation of the $\underset{\sim}{\vec{E}}$-field vector when the thumb points in the direction of propagation.

Polarization of an electric field: **(a)** Linear polarization. **(b)** Circular polarization (left-hand circular polarization if wave propagates in $\hat{\mathbf{z}}$ direction). **(c)** Circular polarization (right-hand if wave propagates in $\hat{\mathbf{z}}$ direction). **(d)** Elliptical polarization.



Polarization of an antenna in a given direction is defined as "the polarization of the wave radiated by the antenna". If the direction is not specified, the direction of maximum gain (see definition later) is understood. Using this definition, the polarization of a dipole antenna is linear and in the same direction as the antenna axis.

Illustration:

or, simply,

It should be pointed out that the earth's presence will affect the antenna pattern, depending on how much the characteristics differ from that of perfect ground.

Note that for linearly polarized waves, maximum reception of power will occur when the receiving antenna is aligned along the direction of polarization (eg. vertical antennas for AM reception; often TV antennas are horizontal; etc.). For circularly polarized waves, orientation of the antenna is not critical as long as its axis is perpendicular to the propagation direction (eg., most FM is circularly polarized).

Circular Polarization Reception:

Mathematical Note on Polarization:

Suppose we have a plane wave travelling in the ${}^+z$ direction and suppose its $\underset{\sim}{\vec{E}}$-field is given by

$$\underset{\sim}{\vec{E}} = \underset{\sim}{E}_x \hat{x} + \underset{\sim}{E}_y \hat{y}$$

with

$\underset{\sim}{E}_x = E_{x_0} \cos(\omega t - kz + \phi_a)$ and $\underset{\sim}{E}_y = E_{y_0} \cos(\omega t - kz + \phi_b)$,

where the amplitudes are obvious and the ϕ's are phase angles. Suppose next that $\phi_b = \phi_a$ or $\phi_b = \phi_a + \pi$. Then

What conditions on the phases and amplitudes of $\underset{\sim}{E}_x$ and $\underset{\sim}{E}_y$ lead to circular polarization of the wave? Prove it! Which is necessary for LHC and which is necessary for RHC polarization? If neither linear nor circular polarization exists, the wave is generally elliptically polarized.

(iii.) Principal E- and H-plane Patterns:

For linearly polarized antennas, the performance may be described in terms of its principal E- and H-plane patters. The principal E-plane is "the plane containing the electric field vector and the direction of maximum radiation". The principal H-plane is "the plane containing the magnetic field vector and the direction of maximum radiation". The E- and H-field patterns in these planes are the respective *principal plane patterns.*

Illustration: Recall, the doughnut-shaped pattern of the current element on page 16.

2. Classes of Radiators

(i.) An isotropic radiator is a "hypothetical lossless antenna having equal radiation in all directions". While no such structure actually exists in nature, it is useful in that the directive properties of realizable antennas may be referenced to this ideal radiator. For example, it is common to find in the literature the term dBi for "dB with respect to isotropic" to indicate how a particular antenna "gain" compares to that of an isotropic radiator (more on "gain" shortly).

(ii) A directional antenna is one which radiates or receives e-m waves more efficiently in some directions than in others. The term is usually applied to antennas whose "directivity" significantly exceeds that of a half-wave dipole (more on "directivity" soon). Diagram (a) on page 17 of this unit shows the pattern of such a directional antenna. The pattern on the bottom of page 16 is non-directional in any *azimuth* plane $\theta =$ constant (for example, the horizontal plane antenna pattern function, $f(\phi),\ \theta = \pi/2$, is a circle in the x-y plane) and directional in any *elevation* plane (with vertical plane pattern given, for example, by $g(\theta),\ \phi =$ constant). Such an antenna is termed *omnidirectional* meaning that it has essentially a non-directional pattern in a given plane (in this case, in azimuth) and a directional pattern in any orthogonal plane (in this case, in elevation) – i.e. "omnidirectional" is a special case of "directional". The pattern on page 17, however, is directional, but NOT omnidirectional.

2.3.2 Radiation Power Density, Radiated Power, and Radiation Resistance

The time-averaged Poynting vector in equation (2.20) is, of course, valid for the far-field of any antenna. That is

$$\vec{\mathcal{P}}_a = \frac{1}{2}\mathcal{R}e\left\{\vec{E} \times \vec{H}^*\right\} \tag{2.39}$$

where $\vec{E}$ and $\vec{H}$ are phasors. The magnitude of this expression is the power density in W/m^2. Thus, to determine the average radiated power, P_r, radiated by the antenna, we simply needed to integrate $\vec{\mathcal{P}}_a$ over a surface *enclosing* the antenna.

$$P_r = \oint_S \vec{\mathcal{P}}_a \cdot d\vec{S} \tag{2.40}$$

where, in general, $d\vec{S}$ is the vector differential area on the surface ($d\vec{S} = \hat{n}dS$ where $\hat{n}$ is the unit normal). In view of (2.39), equation (2.40) may be written

$$P_r = \frac{1}{2}\oint_S \mathcal{R}e\left\{\vec{E} \times \vec{H}^*\right\} \cdot d\vec{S} \tag{2.41}$$

For the infinitesimal dipole, equation (2.22) gave

$$P_r = \frac{\eta k^2 |I_0|^2 \ell^2}{12\pi} \tag{2.42}$$

or given that $k = 2\pi/\lambda$,

$$P_r = \eta\left(\frac{\pi}{3}\right)|I_0|^2\left(\frac{\ell}{\lambda}\right)^2 . \tag{2.43}$$

However, from our usual view of power we know that the radiated power power must also be given by

$$P_r = \frac{1}{2}|I_0|^2 R_r \tag{2.44}$$

where R_r is the so-called *radiation resistance* – i.e. it is the effective resistance which would give rise to power dissipation, P_r, when the current magnitude is $|I_0|$. It is important to realize that R_r is NOT ohmic resistance and P_r is NOT ohmic power dissipation. Equating the RHS of equations (2.43) and (2.44) give for the infinitesimal dipole

$$R_r = \eta \left(\frac{2\pi}{3}\right)\left(\frac{\ell}{\lambda}\right)^2 , \tag{2.45}$$

and, for free space where $\eta = \eta_0 \approx 120\pi$,

$$R_r = 80\pi^2 \left(\frac{\ell}{\lambda}\right)^2 . \tag{2.46}$$

We stipulated for this dipole that, as a rule of thumb, $\ell \leq 0.02\lambda$. Thus, even in the best case (i.e., $\ell = 0.02\lambda$), it is simple to observe that connecting the infinitesimal dipole to a practical transmission line having, say, a $50\,\Omega$ or $75\,\Omega$ characteristic impedance (Z_0) is a very inefficient way to radiate energy:

The gross mismatch with the given Z_0's and thus the inefficiency is obvious. Note that from transmission line deliberations you should be able to conclude that the *reactance* of the infinitesimal dipole is *capacitive*.

The above example serves to illustrate the fact that short antennas (i.e., short compared to a wavelength or *electrically* short) have a very low radiation resistance, low efficiency and thus as we shall see low gain. An *efficient* antenna will have a dimension comparable to a wavelength. Therefore, for example, at AM broadcast frequencies of 500 to 1500 kHz very high towers are required to carry the antennas.

2.3.3 Radiation Intensity, Directivity and Gain

1. Solid Angle

Before discussing the above related quantities, it will be useful to introduce the concept of the *solid angle*. Recall from two-dimensional geometry that the *radian* is defined as the measure of the central angle of a circle which subtends an arc length equal to the radius. Trivially, since the total arc length is $2\pi r$ there are 2π radians in a complete circle. Extending this idea to three dimensions, we define a *steradian* (sr) as the measure of a *solid angle* (Ω) whose vertex is at the centre of a sphere of radius r and which subtends a surface area of r^2 on the sphere.

In general, $\Omega = \dfrac{S}{r^2}$ so when $S = r^2$, $\Omega = 1$ sr. Since, in spherical coordinates, $dS = r^2 \sin\theta d\theta d\phi$,

$$d\Omega = \frac{dS}{r^2} = \sin\theta d\theta d\phi \; . \tag{2.47}$$

Therefore, for the whole sphere,

$$\Omega =$$

Radiation Intensity

The radiation intensity, U, is by definition "the power radiated per unit solid angle". From equation (2.40), the differential radiated power is clearly

$$dP_r = \vec{\mathcal{P}}_a \cdot d\vec{S}$$

over an element of surface $d\vec{S} = dS\hat{r}$. Therefore, Then, the radiation intensity, by definition, is

$$U = \frac{dP_r}{d\Omega} = \frac{1}{2} r^2 \mathcal{R}e \left\{ \vec{E} \times \vec{H}^* \right\} \cdot \hat{r} \tag{2.48}$$

Remembering that the cross product in (2.48) is in $\hat{r}$ direction for the far-field (i.e., $\vec{\mathcal{P}}_a = |\vec{\mathcal{P}}_a|\hat{r}$),

$$\boxed{U = r^2 |\vec{\mathcal{P}}_a|} \tag{2.48a}$$

From equations (2.47) and (2.48), it may also be seen that

$$\tag{2.49}$$

The radiation intensity of the inifinitesimal dipole is left as an exercise.

Of course, in general, U in (2.48) and (2.49) may be a function of θ and ϕ; i.e., $U \equiv U(\theta, \phi)$. However, for an *isotropic* source, U is a constant, say U_0. The power radiated by such a source becomes, from (2.49)

or, equivalently, for an isotropic source,

$$U_0 = \frac{P_r}{4\pi} \tag{2.50}$$

For an *non-isotropic* source, U_0 is the *average* radiation intensity (over all directions).

Finally, we note that a normalized radiation intensity

$$U_n(\theta, \phi) = \frac{U(\theta, \phi)}{U_{\max}} = |f(\theta, \phi)|^2$$

may be used to plot a power pattern $|f(\theta, \phi)|^2$ which has been normalized to unity in the direction of maximum power and $U_{\max}$ is the corresponding maximum radiation intensity. For clarification, it may be noted by way of example that for the infinitesimal dipole, $f(\theta, \phi) = \sin\theta$ and $|f(\theta, \phi)|^2 = \sin^2\theta$ gives the normalized power pattern – we have now quantified what was discussed in a more descriptive way previously for this dipole.

3. Directivity

The *directivity*, D (or $D(\theta,\phi)$ since, it is, in general, direction-dependent) is defined as "the ratio of the power radiated per unit solid angle to the average power radiated per unit solid angle". From the definition of radiation intensity and average radiation intensity and using equations (2.48) and (2.50),

$$D(\theta,\phi) = \frac{U(\theta,\phi)}{U_0} = \frac{dP_r/d\Omega}{P_r/4\pi} \ , \tag{2.51}$$

and the directivity is clearly dimensionless. In an exercise, it will be verified that for the infinitesimal dipole we get, on using (2.48), (2.50) and (2.51) and our earlier results for P_r, that the directivity is given by

$$D(\theta,\phi) = 1.5\sin^2\theta \ . \tag{2.52}$$

4. Efficiency and Gain

If the total input power to an antenna is P_{in} and the radiated power is P_r, then the *radiation efficiency*, or simply efficiency, ε_r, is defined as

$$\varepsilon_r = \frac{P_r}{P_{\text{in}}} \tag{2.53}$$

It must be noted that P_{in} is the actual power delivered to the antenna and NOT the power delivered to the transmission line by the power supply. That is, ε_r does NOT account for impedance mismatch between the supply line and the antenna, rather it accounts for dielectric effects (ohmic resistance etc.) of the antenna and its surroundings. If an antenna is "lossless", $\varepsilon_r = 1$.

The *gain*, $G(\theta,\phi)$, of an antenna is defined as

$$\begin{aligned} G(\theta,\phi) &= 4\pi\frac{\text{radiation intensity}}{\text{input power}} \\ &= 4\pi\frac{dP_r/d\Omega}{P_{\text{in}}} \end{aligned}$$

Therefore,

$$G(\theta, \phi) = \varepsilon_r \frac{dP_r/d\Omega}{P_r/4\pi} \tag{2.54}$$

Using (2.51) and (2.54)

$$G(\theta, \phi) = \varepsilon_r D(\theta, \phi) \tag{2.55}$$

The gain is often incorporated into a parameter called the *effective isotropic radiated power*, EIRP. The EIRP is defined as

$$\mathrm{EIRP} = P_{\mathrm{in}} G_{\mathrm{max}} \tag{2.56}$$

where G_{max} is the maximum gain. The importance of equation (2.56) is that it shows, for a given power requirement, P_{in} can be reduced by increasing antenna gain.

2.3.4 Half-Power Beamwidth

The *half-power beamwidth* is defined as "the angle between the two directions in which the radiation intensity is one-half the maximum as measured in a plane containing the beam maximum". For example, for the small current element discussed earlier, recall that

$$U(\theta, \phi) \propto \sin^2 \theta$$

all other parameters being constant.

If no adjective or descriptor is used with the word "beamwidth", it is generally understood to mean the half-power or 3-dB beamwidth.

2.3.5 Reciprocity

The question may be profitably asked, "Does an antenna in reception mode exhibit the same characteristics (i.e., the same antenna pattern parameters) as in transmit mode?" The answer lies in the concept of *reciprocity*.

Circuit Analogy:

Consider a two-port network consisting of linear, bilateral (i.e., same looking from both ports), lumped elements.

The reciprocity theorem states that if one places a constant voltage (or current) generator across one port and places a current (or voltage) meter across the other port, makes observations of the meter readings and then interchanges the location of the source and the meter, the meter reading will be unchanged (in fact, this is true for any pair of ports in a multiport network).

In a similar fashion, consider the following antenna setups:

By reciprocity,

$$\text{if } \; V_B = V_A, \; \text{ then } I_A = I_B \; . \tag{2.57}$$

Also, defining the transfer impedance in Case 1 as $Z_{AB} = \dfrac{V_A}{I_B}$ and in Case 2 as $Z_{BA} = \dfrac{V_B}{I_A}$, reciprocity says that $Z_{AB} = Z_{BA}$.

Let's prove (2.57) by using the circuit analogy – i.e., let's start by representing the antennas and intervening space by a two-port network of linear, passive, and bilateral impedances. Insofar as the input voltage and output current are concerned, such a network may be reduced to an equivalent T-section as shown:

<u>Case 1.</u>

<u>Case 2.</u>

For Case 1,

Substituting (2.58) into (2.59) relates I_B to V_A and the impedances as

$$I_B = \frac{V_A Z_3}{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3} \,. \tag{2.60}$$

Interchanging the source and current meter (Case 2) gives in exactly the same way

$$I_A = \frac{V_B Z_3}{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3} \,. \tag{2.61}$$

We see from (2.60) and (2.61) that if $V_A = V_B$ then $I_A = I_B$ and the reciprocity theorem is verified. REMEMBER, the space between the antennas must NOT exhibit directional properties for this to hold.

The idea of reciprocity becomes important in considering mutual impedances due to coupling effects experienced by antennas in close proximity. We shall leave the problems of both *self-* and *mutual-*impedance for a later portion of the course. To indicate the magnitude of this problem (in reality), however, it may be pointed out that the mutual impedance development involves the "solution" of a pair of coupled integral equations.

2.3.5.1 More on Reciprocity

While magnetic current densities do not exist in nature (i.e. there are no magnetic charges, as far as we know), it is sometimes useful to postulate such entities (this will be discussed in more detail later in the course). If we allow for a magnetic current density $\vec{M}$ (analogous to current density $\vec{J}$), the Maxwell equations may be cast as

$$\vec{\nabla} \times \vec{E} = -j\omega\mu\vec{H} - \vec{M} \tag{2.62}$$

$$\vec{\nabla} \times \vec{H} = j\omega\epsilon\vec{E} + \vec{J} \tag{2.63}$$

$$\vec{\nabla} \cdot \vec{D} = \rho_v \tag{2.64}$$

$$\vec{\nabla} \cdot \vec{B} = \rho_m \tag{2.65}$$

where ρ_m is the "ficticious" magnetic charge density.

The Lorentz Reciprocity Theorem

1. Differential Form

Consider a linear isotropic medium (not necessarily homogeneous) containing two sets of sourses (electric and magnetic current densities) $\vec{J}_1$, $\vec{M}_1$ and $\vec{J}_2$, $\vec{M}_2$. These sources are allowed to radiate, together or singly, giving rise to fields $\vec{E}_1$, $\vec{H}_1$ and $\vec{E}_2$, $\vec{H}_2$, respectively. We will now show the validity of the following statement referred to as the *Lorentz Reciprocity Theorem* (differential form):

$$-\vec{\nabla} \cdot \left[\left(\vec{E}_1 \times \vec{H}_2 \right) - \left(\vec{E}_2 \times \vec{H}_1 \right) \right] = \left(\vec{E}_1 \cdot \vec{J}_2 \right) + \left(\vec{H}_2 \cdot \vec{M}_1 \right) - \left(\vec{E}_2 \cdot \vec{J}_1 \right) - \left(\vec{H}_1 \cdot \vec{M}_2 \right) \tag{2.66}$$

For the first set of sources and fields

$$\vec{\nabla} \times \vec{E}_1 = -j\omega\mu \vec{H}_1 - \vec{M}_1 \tag{2.67}$$

$$\vec{\nabla} \times \vec{H}_1 = j\omega\epsilon \vec{E}_1 + \vec{J}_1 \tag{2.68}$$

and for the second set

$$\vec{\nabla} \times \vec{E}_2 = -j\omega\mu \vec{H}_2 - \vec{M}_2 \tag{2.69}$$

$$\vec{\nabla} \times \vec{H}_2 = j\omega\epsilon \vec{E}_2 + \vec{J}_2 \tag{2.70}$$

where we have simply used (2.62) and (2.63) within the given medium for which the permittivity and permeability are ϵ and μ, respectively.

Taking the dot product of (2.67) and (2.70) with $\vec{H}_2$ and $\vec{E}_1$, respectively, gives

$$\begin{aligned}\vec{H}_2 \cdot \left(\vec{\nabla} \times \vec{E}_1\right) &= -j\omega\mu \left(\vec{H}_2 \cdot \vec{H}_1\right) - \left(\vec{H}_2 \cdot \vec{M}_1\right) && (2.71)\\ \vec{E}_1 \cdot \left(\vec{\nabla} \times \vec{H}_2\right) &= j\omega\epsilon \left(\vec{E}_1 \cdot \vec{E}_2\right) + \left(\vec{E}_1 \cdot \vec{J}_2\right) && (2.72)\end{aligned}$$

Subtract (2.71) from (2.72) to get

$$\vec{E}_1 \cdot \left(\vec{\nabla} \times \vec{H}_2\right) - \vec{H}_2 \cdot \left(\vec{\nabla} \times \vec{E}_1\right) = j\omega\epsilon \left(\vec{E}_1 \cdot \vec{E}_2\right) + j\omega\mu \left(\vec{H}_2 \cdot \vec{H}_1\right) + \left(\vec{E}_1 \cdot \vec{J}_2\right) + \left(\vec{H}_2 \cdot \vec{M}_1\right) \ . \tag{2.73}$$

Recalling that for vectors $\vec{A}$ and $\vec{B}$, in general,

$$\vec{B} \cdot \left(\vec{\nabla} \times \vec{A}\right) - \vec{A} \cdot \left(\vec{\nabla} \times \vec{B}\right) = \vec{\nabla} \cdot \left(\vec{A} \times \vec{B}\right) \ ,$$

(2.73) may also be written as

$$\vec{\nabla} \cdot \left(\vec{H}_2 \times \vec{E}_1\right) = -\vec{\nabla} \cdot \left(\vec{E}_1 \times \vec{H}_2\right) = \text{RHS of (2.73)} \ . \tag{2.74}$$

Next, taking the dot product of (2.68) and (2.69) with $\vec{E}_2$ and $\vec{H}_1$, respectively, and carrying out the above procedure yields

$$\vec{\nabla} \cdot \left(\vec{H}_1 \times \vec{E}_2\right) = -\vec{\nabla} \cdot \left(\vec{E}_2 \times \vec{H}_1\right) = j\omega\epsilon \left(\vec{E}_2 \cdot \vec{E}_1\right) + j\omega\mu \left(\vec{H}_1 \cdot \vec{H}_2\right) + \left(\vec{E}_2 \cdot \vec{J}_1\right) + \left(\vec{H}_1 \cdot \vec{M}_2\right) \ . \tag{2.75}$$

Subtracting (2.75) from (2.74) with an eye on (2.73) gives

$$-\vec{\nabla} \cdot \left[\left(\vec{E}_1 \times \vec{H}_2\right) - \left(\vec{E}_2 \times \vec{H}_1\right)\right] = \left(\vec{E}_1 \cdot \vec{J}_2\right) + \left(\vec{H}_2 \cdot \vec{M}_1\right) - \left(\vec{E}_2 \cdot \vec{J}_1\right) - \left(\vec{H}_1 \cdot \vec{M}_2\right) \tag{2.76}$$

as required. This is the differential form of the *Lorentz Reciprocity Theorem.*

2. Integral Form

The integral form of (2.76) follows from the divergence theorem. Recall

$$\int_{\text{vol}} \left(\vec{\nabla} \cdot \vec{A}\right) dv = \oint_S \vec{A} \cdot d\vec{S} \ . \tag{2.77}$$

From (2.76) we identify

$$\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot - \left[\left(\vec{E}_1 \times \vec{H}_2\right) - \left(\vec{E}_2 \times \vec{H}_1\right)\right]$$

or, equivalently,

$$\vec{\nabla} \cdot \vec{A} = \left(\vec{E}_1 \cdot \vec{J}_2\right) + \left(\vec{H}_2 \cdot \vec{M}_1\right) - \left(\vec{E}_2 \cdot \vec{J}_1\right) - \left(\vec{H}_1 \cdot \vec{M}_2\right)$$

and substituting into (2.77) we may write

$$-\oint_S \left[\left(\vec{E}_1 \times \vec{H}_2\right) - \left(\vec{E}_2 \times \vec{H}_1\right)\right] \cdot d\vec{S} = \int_{\text{vol}} \left[\left(\vec{E}_1 \cdot \vec{J}_2\right) + \left(\vec{H}_2 \cdot \vec{M}_1\right) - \left(\vec{E}_2 \cdot \vec{J}_1\right) - \left(\vec{H}_1 \cdot \vec{M}_2\right)\right] dv \, . \tag{2.78}$$

This is the integral form of the *Lorentz Reciprocity Theorem*.

The Dipole Probe

It is not too hard to argue that on observing the fields in the far field

$$\int_{\text{vol}} \left[\left(\vec{E}_1 \cdot \vec{J}_2\right) - \left(\vec{H}_1 \cdot \vec{M}_2\right)\right] dv' = \int_{\text{vol}} \left[\left(\vec{E}_2 \cdot \vec{J}_1\right) - \left(\vec{H}_2 \cdot \vec{M}_1\right)\right] dv' \, . \tag{2.79}$$

Consider the second source to be a Hertzian dipole (i.e. $\vec{M}_2 = 0$) whose current density can be specified by

$$\vec{J}_2 = \delta(x - x_P)\delta(y - y_P)\delta(z - z_P)\vec{p}$$

where δ is the Dirac delta function and $\vec{p}$ is the vector length of the dipole. Then

$$\text{LHS of (2.79)} = \int_{\text{vol}} \left(\vec{E}_1 \cdot \vec{J}_2\right) dv' = \vec{E}_1(x_P, y_P, z_P) \cdot \vec{p} \, .$$

Thus,

$$\vec{E}_1(x_P, y_P, z_P) \cdot \vec{p} = \int_{\text{vol}} \left[\left(\vec{E}_2 \cdot \vec{J}_1\right) - \left(\vec{H}_2 \cdot \vec{M}_1\right)\right] dv' \tag{2.80}$$

where "vol" contains $\vec{J}_1$ and $\vec{M}_1$. Now, since $\vec{E}_2$ and $\vec{H}_2$ at (x_1, y_1, z_1) (a point in "vol") are the Hertzian dipole fields, and are known, $\vec{E}_1$ at the dipole probe position (x_P, y_P, z_P) can be calculated knowing the sources $\vec{J}_1$ and $\vec{M}_1$.

Exercise: Use (2.80) to show that the far-field of any finite *electric* current distribution in free space can have no radial component.

2.4 A Small Current Loop

Just as the infinitesimal dipole is a good starting point for discussing linear wire antennas, so the small circular current loop will provide an introduction to loop antennas in general. [In this course, we will not consider the general case of larger loops]. This section addresses the far-field only.

Consider the following geometry for a small current loop in the x-y-plane.

Remember that the primed coordinates are used for the source points. Here, the loop radius is r_0, $R = |\vec{r} - \vec{r}\,'|$, and $d\vec{\ell}\,' = r_0 d\phi' \hat{\phi}$. The transformation from cylindrical to Cartesian coordinates (see Appendix VII of text) gives for this last quantity

$$d\vec{\ell}\,' =$$

From equation (1.40), we have the general form of the required vector potential as

$$\vec{A}(\vec{r}) = \int_{v'} \frac{\mu \vec{J}(\vec{r}\,') e^{-jk|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r} - \vec{r}\,'|} dv' \tag{2.81}$$

In the case of the small planar loop, we replace $\vec{J}(\vec{r}\,')dv'$ with $I_0 d\vec{\ell}\,' = I_0 r_0 d\phi' \hat{\phi}$ so that

$$I_0 d\vec{\ell}\,' = I_0 r_0 d\phi' \left[-\sin\phi' \hat{x} + \cos\phi' \hat{y}\right] \; . \tag{2.82}$$

Furthermore, we note:

Now, $R = |\vec{r} - \vec{r}'| = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$ and, based on *,

$$R = |\vec{r} - \vec{r}'| =$$

Therefore, it may be easily seen that

$$R = [r^2 + r_0^2 - 2rr_0 \sin\theta(\cos\phi\cos\phi' + \sin\phi\sin\phi')]^{1/2} \tag{2.83}$$

In (2.81), we notice a $\frac{1}{R}$ factor in the amplitude and, since $r_0 \ll r$ for the small loop, we may as well replace R by r for this factor. However, as with the infinitesimal dipole, we must be very careful about approximations in the phase term, e^{-jkR}. We agree to the following: $r_0^2 \ll r^2$ (even in terms of the phase, this will be OK). However, the third term in (2.83) will have to be considered closely because, while $r_0^2 \ll r^2$, the same order does not exist for rr_0, r itself being very large. Using the binomial expansion on the remaining part of (2.83), one obtains (after eliminating r_0^2)

$$R \approx r[1 - \frac{r_0}{r}\sin\theta(\cos\phi\cos\phi' + \sin\phi\sin\phi')] \ ,$$

and

$$kR \approx kr - kr_0\sin\theta(\cos\phi\cos\phi' + \sin\phi\sin\phi') \ ,$$

which implies

$$e^{-jkR} = e^{-jkr}e^{+jkr_0 \sin\theta(\cos\phi\cos\phi' + \sin\phi\sin\phi')} \ . \tag{2.84}$$

Also, for the small loop $r_0 \ll \lambda$. Therefore, $kr_0 = \frac{2\pi r_0}{\lambda} \ll 1$. Then, we know from the Maclaurin series that for $f(x) \ll 1$,

$$e^{f(x)} \approx \qquad\qquad\qquad\qquad .$$

On this basis, (2.84) becomes

$$e^{-jkR} = e^{-jkr}\left[1 + jkr_0 \sin\theta(\cos\phi\cos\phi' + \sin\phi\sin\phi')\right] \ . \tag{2.85}$$

Noting the several constants (I_0, r_0, etc.) associated with this analysis and using (2.82) and (2.85) along with the approximation in amplitude of $R \approx r$, (2.81)may be written as

$$\vec{A}(\vec{r}) =$$

$$=$$

$$\tag{2.86}$$

Since

$$\vec{A}(\vec{r}) = \tag{2.87}$$

Again, recall

We finally have for the vector potential of the small current loop

$$\vec{A}(\vec{r}) = \frac{jk\mu(I_0\pi r_0^2)}{4\pi r}e^{-jkr}\sin\theta\hat{\phi} \tag{2.88}$$

In passing, we note that $(I\pi r_0^2) \equiv$ (current times loop area) is the magnitude of a quantity referred to as the *magnetic dipole moment*, $\vec{M}$, of the small current loop and it points perpendicular to the loop in the sense given by the right-hand rule as depicted (i.e., with fingers of the right hand curled in direction of current, $\vec{M}$ points in the direction of the thumb):

Here, then, $\vec{M} = I_0\pi r_0^2\hat{z}$.

$\vec{H}$-Field

Now, with the vector potential established, we may proceed as usual to find the fields. Using (2.88) and recalling that

$$\vec{H} = \frac{1}{\mu}\vec{\nabla}\times\vec{A}\ ,$$

it is easily seen that

$$\vec{H} = -\frac{1}{\mu r}\frac{\partial}{\partial r}[rA_\phi]\hat{\theta}\ .$$

Then (DO THIS)

$$\vec{H} = -\frac{k^2(I_0\pi r_0^2)\sin\theta}{4\pi r}e^{-jkr}\hat{\theta} \tag{2.89}$$

$\vec{E}$-Field Finally, using the far-field, source free form of (1.20) it may be easily established (DO IT) that

$$\vec{E} = \frac{\eta k^2(I_0\pi r_0^2)\sin\theta}{4\pi r}e^{-jkr}\hat{\phi} \tag{2.90}$$

Poynting Vector, Radiation Intensity, Power and Resistance

Without doing the calculations here (BUT THEY SHOULD BE DONE), we have:

Time-averaged Poynting Vector:

$$\vec{\mathcal{P}}_a = \frac{1}{2}\mathcal{R}e\left\{\vec{E}\times\vec{H}^*\right\} = ? \tag{2.91}$$

Radiation Intensity: $U = r^2|\vec{\mathcal{P}}_a| = ?$

Radiated Power:

$$P_r = \oint_S \vec{\mathcal{P}}_a \cdot d\vec{S} = \int_0^{2\pi} \int_0^{\pi} U \sin\theta d\theta d\phi = ? \tag{2.92}$$

Radiation Resistance:

$$R_r = \frac{2P_r}{|I_0|^2} = 320\pi^6 \left(\frac{r_0}{\lambda_0}\right)^4 , \tag{2.93}$$

this last expression being given for free-space operation. We note that a quick check of the initial power expression or of equation (2.93) itself shows that the radiation resistance can be written in terms of the loop area, A_L as

$$R_r = 31,171 \left(\frac{A_L^2}{\lambda_0^4}\right)$$

for free-space operation (λ_0 is the free-space wavelength). Finally, if N turns rather than a single loop were used then R_r increases by a factor of N^2 so that for free space

$$R_r = 320\pi^6 N^2 \left(\frac{r_0}{\lambda_0}\right)^4 . \tag{2.94}$$

This is because the magnetic dipole moment for N turns increases by a factor of N as compared to a single loop – note that the cross product in $\vec{\mathcal{P}}_a$ will produce an N^2.

The radiation resistance, as usual, is to be distinguished from ohmic loss. If the loop is not lossless, but highly conducting, the *ohmic resistance* can be determined via the skin depth as discussed in elementary e-m texts. Additionally, for several loops, there is an ohmic term due to a phenomenon termed the "proximity effect". The latter is documented in graphical form in the literature and may be found in the Balanis reference.

As for the inifinitesimal dipole, the radiation resistance R_r for the small loop is very small and such a loop is not generally suited as a transmitting antenna. Small loops may be used as receiving antennas where signal level is large (eg. multiturn loops are used in portable AM radio reception and single turns are used in pagers) or where noise limitations are such that improving antenna efficiency does not necessarily give better reception.

Also, we note that the antenna efficiency, ε_r, given by equation (2.53) may be written (not just for circular loops)

$$\varepsilon_r = \frac{R_r}{R_r + R_L} \tag{2.95}$$

where R_L are losses due to conduction–dielectric effects and R_r is the radiation resistance.

Finally, we note that for this small loop $\vec{H} = H_\theta \hat{\theta}$ and $\vec{E} = E_\phi \hat{\phi}$ (equations (2.89) and (2.90)), while for the infinitesimal dipole $\vec{E} = E_\theta \hat{\theta}$ and $\vec{H} = H_\phi \hat{\phi}$ (equations (2.16) and (2.17)). However, in both cases, $\mathcal{P}_a$ will vary as $\sin^2\theta$ so that the radiation patterns of the loop have the same general characteristics as the dipole. We note for the loop that the nulls occur at $\theta = 0, \pi$ – i.e., along the axis perpendicular to the loop. See reverse side for diagrams.