Unit 1

Relevant Electrostatics and Magnetostatics (Old and New)

The whole of classical electrodynamics is encompassed by a set of coupled partial differential equations (at least in one form) bearing the name "Maxwell's Equations". On the basis of material already encountered and some new concepts, it is the purpose of this course to develop these equations and to make some elementary applications. To facilitate these intentions, we shall briefly review a little of the prerequisite electrostatic and magnetostatic material and, in so doing, revisit some of the necessary mathematics. It should be remembered that much of what now may be written as concise mathematical formalism was, in fact, often developed initially by painstaking experimental investigation. We shall seek to not stray too far from the physical significance and practicalities of the results "derived". For the level and length of this course, it will not be possible to explore in detail the scope of the usefulness of the equations considered. In fact, in all but the simplest cases, solutions in closed form do not exist. It will quickly become apparent, especially in future studies based on this course, that numerical methods and approximation techniques are essential in most engineering applications of Maxwell's equations. Still, there will be plenty of "interesting" problems which may be tackled with the arsenal of tools developed in this and earlier terms.

A Few Preliminary Observations on Notation

(1) We shall use $\vec{r}$ to represent the position vector in space for some observation point

(relative to an origin) and $\vec{r}'$ to indicate the position of <u>sources</u> (relative to an origin).
(2) Based on note (1), $\vec{r} - \vec{r}'$ will thus be the displacement vector from the source to the observation position. This vector will assume various labels throughout the course (eg., R, R_{12}, etc.) as may be convenient.

<u>Illustration:</u>

(3) The text uses bold letters to represent vector quantities. We shall use vector signs $\vec{}$ or $\rightharpoonup$ (eg., $\vec{E}$ or $\overset{\rightharpoonup}{E}$ ≡ electric field intensity). We'll use $\hat{}$ to indicate unit vectors (eg., $\hat{x}$ ≡ unit vector in the x direction).

Reference should be made to the text for various coordinate systems which may be encountered. Also, the vector operators, divergence, gradient and curl ($\vec{\nabla}\cdot$, $\vec{\nabla}$ and $\vec{\nabla}\times$) are found at the back of the text, while various vector operator identities are located in Appendix A.3.

1.1 Electrostatics

For the sake of completeness and for future reference, we shall write down the following previously encountered results.

<u>Coulomb's Law:</u>

With reference to the accompanying diagram, the force, $\vec{F}_{21}$, on charge Q_2 due to charge Q_1 is given by

$$\begin{aligned} \vec{F}_{21} = -\vec{F}_{12} &= \\ &= \end{aligned} \tag{1.1}$$

where we have assumed that the charges are in free space and ϵ_0 (=8.85×10^{-12} F/m) is the *permittivity* of free space. The definition of unit vector is obvious.

Electric Field Intensity:

The electric field intensity, $\vec{E}$, is by definition the force per unit charge on a small positive test charge, q_t, brought into the field and is in the direction of the force; i.e.

$$\vec{E} = \frac{\vec{F}}{q_t} \tag{1.2}$$

The unit is N/C or, more conveniently, volt/metre (V/M). For a volume charge distribution with charge density $\rho_v(\vec{r}')$,

$$\vec{E}(\vec{r}) = \tag{1.3}$$

Note that dv' is the elemental volume at $\vec{r}'$. Clearly, the integral sums the effects of all the charges of the form $\rho_v(\vec{r}')dv'$. In Cartesian coordinates, $dv' = dx'dy'dz'$. This integral can be "nasty" and is usually explicitly carried out only for simple cases. (There are often better ways of finding $\vec{E}$, eg., Gauss' Law).

Electric Flux Density and Gauss' Law:

Gauss' Law states that the electric flux, Ψ, measured in coulombs (C), passing through any closed surface is equal to the total charge enclosed by that surface.

The electric flux density, $\vec{D}$, is measured in C/m^2.

$$\Psi = Q = \oint_S \vec{D} \cdot d\vec{S} = \int_{\text{vol}} \rho_v dv \tag{1.4}$$

Note that

$$d\vec{S} = dS\hat{S}$$

where $\hat{S}$ is the outward unit normal to the elemental surface dS. For example, in cartesian coordinates, $dS_z = dxdy$.

We also note a so-called *constitutive relation* between $\vec{E}$ and $\vec{D}$. For free space it is

$$\vec{D} = \epsilon_0 \vec{E} \tag{1.5}$$

Gauss' Integral Theorem (or the Divergence Theorem):

This theorem states that

$$\oint_S \vec{D} \cdot d\vec{S} = \int_{\text{vol}} \vec{\nabla} \cdot \vec{D} dv \qquad (1.6)$$

$\vec{\nabla} \cdot \vec{D}$ is the divergence of $\vec{D}$ throughout the volume. Equation (1.6) is a general statement for vector fields, not just $\vec{D}$-fields. Note that (1.4) and (1.6) give

$$\vec{\nabla} \cdot \vec{D} = \rho_v \qquad (1.7)$$

which happens to be the "point form" of one of Maxwell's equations alluded to earlier.

Aside: Recall

Equation (1.7) holds for time-varying fields also, and it will be used in this context later in the course.

Work, W, Done *on* a Charge, Q, Moved in an $\vec{E}$–Field:

$$W = -Q \int_B^A \vec{E} \cdot d\vec{L} \qquad (1.8)$$

where $d\vec{L}$ is a differential displacement vector along the path, C, from B to A as illustrated.

Not surprisingly, only the component of $\vec{E}$ along the path contributes to energy expenditure or work done.

Potential Difference, V_{AB}, and Potential at a Point and Potential Gradient:

From equation (1.8), we define the potential difference between points A and B as

$$V_{AB} = \frac{W}{Q} = -\int_B^A \vec{E} \cdot d\vec{L} = V_A - V_B \qquad (1.9)$$

where $V_A \equiv$ potential at point A, $V_B \equiv$ potential at point B. The potential at a point is the potential difference measured with respect to some arbitrarily chosen zero reference point. For example, if $V_\infty = 0$, then V_A is the work done per unit charge in moving the charge from ∞ to point A.

Conservative Field:

Note that, by definition, a conservative field is one which satifies the condition " a closed line integral of the field is zero". For example, the electrostatic field, $\vec{E}$, is conservative so

$$\oint_C \vec{E} \cdot d\vec{L} = 0 \tag{1.10}$$

Note that the path is closed. Equation (1.10) is not valid for time-varying fields – more later!!

Also, recall, importantly,

$$\vec{E} = -\vec{\nabla} V \tag{1.11}$$

which provides a convenient way of finding the (vector) electric field, $\vec{E}$, if the (scalar) potential field, V, is known.

Continuity of Current:

Consider a current density, $\vec{J}$, measured in amperes per square metre (A/m^2). The principle of charge conservation leads to the point form of the equation of continuity of current as

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \tag{1.12}$$

Boundary Conditions:

(1) Perfect Conductor

Recall that within a *perfect* conductor no free charge and no electric field may exist. Using this fact, it is easily argued from equation (1.10) that at the surface between a perfect conductor and free space

$$E_t = 0 \Longrightarrow D_t = 0$$

and (1.13)

$$D_N = \rho_s \quad (\text{or } E_N = \frac{\rho_s}{\epsilon_0})$$

where $t \equiv$ tangential, $N \equiv$ normal, and ρ_s is a surface charge density (charge may exist on the conductor surface).

(2) Perfect Dielectric

Now, in general, $\vec{E}$ may be non-zero on both sides of the boundary. Again, arguing from equation (1.10), it may be verified that for the case at hand

$$E_{t_1} = E_{t_2}$$

and (1.14)

$$D_{N_1} = D_{N_2} \quad (\text{or } E_{N_1} = \frac{\epsilon_2}{\epsilon_1} E_{N_2})$$

with ϵ_1 and ϵ_2 being the material permittivities.

Poisson's and Laplace's Equations:

Finally, in the electrostatics portion of the previous course, the following important results were derived from equations (1.7), (1.5), and (1.11):

(1) Poisson's Equation:

$$\vec{\nabla} \cdot \vec{\nabla} V = -\frac{\rho_v}{\epsilon} \tag{1.15}$$

where ρ_v is the usual volume charge density and ϵ is the permittivity of the region under consideration.

(2) Laplace's Equation:

A special case of (1.15) occurs in a region for which $\rho_v = 0$. The result, known as Laplace's equation is clearly given by

$$\vec{\nabla} \cdot \vec{\nabla} V = 0$$

commonly abbreviated as

$$\vec{\nabla}^2 V = 0 \tag{1.16}$$

1.2 Magnetostatics

Electric charges give rise to electric fields; electric currents are a (one) source of *magnetic* fields. We shall briefly review some of the important properties of steady magnetic fields which arise from steady (dc) currents. Recall the following:

Biot-Savart Law:

The Biot-Savart law is the magnetic field counterpart (sort of) to Coulomb's law. This law, which relates the *magnetic field intensity*, $\vec{H}$, in amperes per metre (A/m) to a source current, I, may be written in integral form as

(1.17)

This equations relates to the illustration below.

Illustration:

Clearly, from the cross product, the $\vec{H}$ field element ($d\vec{H}$) produced by the current, I, flowing along the differential displacement, $d\vec{L}$, of the closed contour, C, is perpendicular to the plane containing $d\vec{L}$ and the unit vector $\hat{a}_R$ from $d\vec{L}$ (i.e. the source region) to the point of observation of the $\vec{H}$ field. R is simply the distance from $d\vec{L}$ to the place where $\vec{H}$ is observed. The right hand rule for the cross product gives the orientation of the contribution, $d\vec{H}$, due to the current in element $d\vec{L}$.

Equation (1.17) may also be written for surface current densities $\vec{K}$ in A/m or (volume) current density $\vec{J}$ in A/m^2 as

(1.18)

Illustration:

or

(1.19)

To emphasize and illustrate the importance of coordinate system representation of these expressions consider (1.19).

Illustration:

Therefore, (1.19) becomes

(1.20)

Again, note that the primed coordinates represent the source position and the unprimed coordinates represent the field (observation) position. Comparing equation (1.20) (i.e. the Biot-Savart law for the magnetic field) with equation (1.3) (i.e. Coulomb's law for the electric field), the similarities and differences are obvious: (1) Both are inverse square laws, but (2) $\vec{E}$ is in the direction of $\hat{a}_R$ while $\vec{H}$ is perpendicular to both the source vector, $\vec{J}$, and $\hat{a}_R$.

Ampère's Circuital Law:

An analogy to Gauss' law for electrostatics is Ampère's circuital law for magnetostatics. Instead of "talking about" electric flux density and charge enclosed by a surface, S, we consider a steady magnetic field intensity and a current, I_e, enclosed by a contour, C. **Ampère's law** states that the line integral of $\vec{H}$ around any closed path is exactly equal to the direct current, I_e, enclosed by the path. That is

$$\oint_C \vec{H} \cdot d\vec{L} = I_e \ . \tag{1.21}$$

Illustrations: (1) Current Filament:

(2) "Thick" dc-Current-Carrying Wire:

Note, in general, that

$$I_e = \int_S \vec{J} \cdot d\vec{S} \tag{1.22}$$

(Similar ideas exist for surface current density, $\vec{K}$).

If it is possible to choose a contour such that (1) $\vec{H}$ is everywhere tangential to the contour and (2) $\vec{H}$ is constant along the contour, equation (1.21) (Ampère's circuital law) may be used to find $\vec{H}$ when I_e is known. For this purpose, the nature of $\vec{H}$ may be observed from the Biot-Savart law (without actually doing the integration) and symmetry arguments. We shall shortly see that Ampère's circuital law implies the Biot-Savart law (and vice versa).

Stokes' Theorem:

For a vector field $\vec{A}$ and a closed contour, C, surrounding a surface S,

(1.23)

From (1.21), (1.22) and (1.23), it is clear that Ampère's law in point form may be written down as

$$\vec{\nabla} \times \vec{H} = \vec{J} \tag{1.24}$$

[Recall that for electrostatics we had $\oint_C \vec{E} \cdot d\vec{L} = 0$ (for any closed C) which leads to $\vec{\nabla} \times \vec{E} = 0$ – this is definitely NOT TRUE for time-varying fields. Likewise, we shall see that (1.24) will have to be modified to include time-varying characteristics.]

Magnetic Flux Density:

Another of the so-called constitutive relationships states that the magnetic flux density, $\vec{B}$, measured in webers per square metre (Wb/m^2) or tesla (T) is related to the magnetic field intensity, $\vec{H}$, in A/m^2 via the expression

$$\vec{B} = \mu \vec{H}$$

where μ, in henrys per metre (H/m) is referred to as the magnetic *permeability* of the medium in which $\vec{H}$ exists. This parameter is a "description" of the magnetic nature of the medium (more later). For free space,

$$\vec{B} = \mu_0 \vec{H} \tag{1.25}$$

where $\mu_0 = 4\pi \times 10^{-7}$ H/m. Note that "magnetic flux", Φ, in webers is given by analogy to equation (1.4) as

$$\Phi = \oint_S \vec{B} \cdot d\vec{S} = 0$$

since no isolated magnetic "charge" (pole) has been found. Equation (1.6), on replacing $\vec{D}$ with $\vec{B}$, gives

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1.26}$$

We may now summarize the set of equations which describes all non-time-varying electromagnetics:

Point Form:

$$
\begin{aligned}
\vec{\nabla} \times \vec{E} &= 0 \\
\vec{\nabla} \times \vec{H} &= \vec{J} \\
\vec{\nabla} \cdot \vec{D} &= \rho_v \\
\vec{\nabla} \cdot \vec{B} &= 0
\end{aligned}
$$

Integral Form:

$$
\begin{aligned}
\oint_C \vec{E} \cdot d\vec{L} &= 0 \\
\oint_C \vec{H} \cdot d\vec{L} &= I_e \\
\oint_S \vec{D} \cdot d\vec{S} &= Q \\
\oint_S \vec{B} \cdot d\vec{S} &= 0
\end{aligned}
$$

We note too that a vector field is uniquely specified by its divergence and curl. Also, given any vector field $\vec{A}$

(1) $\vec{\nabla} \cdot \vec{A} = 0$ implies that the field is *solenoidal* – i.e., no source or sink in regions where this is true;

(2) $\vec{\nabla} \times \vec{A} = 0$ implies that the field is *irrotational*.

Relevant Scalar and Vector Potentials:

Scalar Potential

We have already seen the scalar potential field, V, associated with the electric field and noted the simplicity with which $\vec{E}$ may be found when V is known – i.e., $\vec{E} = -\vec{\nabla} V$ (see equation (1.11)). In fact using what we already know about a conservative $\vec{E}$ field (i.e., $\vec{\nabla} \times \vec{E} = 0$) and Gauss' law in point form ($\vec{\nabla} \cdot \vec{D} = \rho_v$) it is easy to derive Poisson's equation (see equation (1.15)). Equation (1.15) has a solution given by equation (18), p. 96 of the text. As we have said, if V can be determined, $\vec{E}$ follows immediately from (1.11).

In summary, then, starting with

(Conservative electrostatic field)

(Gauss' law)

(Constitutive relationship)

and choosing $\vec{E} = -\vec{\nabla}V$, we may arrive at Poisson's equation, $\vec{\nabla}^2 V = -\frac{\rho_v}{\epsilon_0}$, from which, as noted above,

$$V(\vec{r}) = \int_{v'} \frac{\rho_v(\vec{r}')dv'}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} \tag{1.27}$$

Vector Potential

A similar idea can be constructed for finding the magnetic field, $\vec{H}$, from a so-called *vector potential* which we shall label $\vec{A}$. Consider this time

$\vec{\nabla} \cdot \vec{B} = 0$ (1.26) (Gauss' Law – no magnetic poles)

$\vec{\nabla} \times \vec{H} = \vec{J}$ (1.24) (Ampère's Law)

$\vec{B} = \mu_0 \vec{H}$ (1.25) (Constitutive Relationship)

It may be recalled (see Appendix A.3 of text) that for any vector $\vec{W}$,

(1.28)

Thus, we choose a "vector potential" $\vec{A}$ defined by

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{1.29}$$

which by (1.28) is a solution of (1.26). We know from (1.25) and (1.20) (the Biot-Savart law) that $\vec{B}$ depends on the (assumed known) source current density, $\vec{J}$. Thus, if we are able to relate $\vec{A}$ to $\vec{J}$, (1.29) will yield $\vec{B}$. [By the way, is $\vec{A}$ unique?]

Using (1.25) in (1.29) and substituting into (1.24), we get

(1.30)

a second-order partial differential equation relating our chosen vector potential to source current density, $\vec{J}$. We wish to obtain a solution for $\vec{A}$ from (1.30) so that it may be used in (1.29) to find $\vec{B}$ or $\vec{H}$.

Again using Appendix A.3 of the text, we observe that

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$$

where

$$\vec{\nabla}^2 \vec{A} = \frac{\partial^2 \vec{A}}{\partial x^2} + \frac{\partial^2 \vec{A}}{\partial y^2} + \frac{\partial^2 \vec{A}}{\partial z^2}$$

(in cartesian coordinates). Using this identity in (1.30) yields

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}\,. \tag{1.31}$$

Let us now assume that, for the case of magnetostatics, $\vec{A}$ is *solenoidal*, i.e.

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{1.32}$$

so that $\vec{A}$ is uniquely specified by (1.29) and (1.32). [We shall address this assumption in the near future.] Based on this requirement, (1.31) reduces to

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J} \tag{1.33}$$

Comparing (1.33) with Poisson's equation gives us some insight as to why $\vec{A}$ is called a vector potential. The solution of (1.33) may be thus patterned after (1.27) and we write it down directly as

$$\vec{A}(\vec{r}) = \int_{v'} \frac{\mu_0 \vec{J}(\vec{r}\,')dv'}{4\pi\,|\vec{r} - \vec{r}\,'|}\,. \tag{1.34}$$

Similarly, in terms of surface current density, $\vec{K}$,

(1.35)

and for line currents, I,

(1.36)

Note that while $\vec{H}$ is perpendicular to the plane defined by $Id\vec{r}\,'$ and the observation point, $\vec{A}$ is in the same direction as I. We note that (1.34)–(1.36), unlike the Biot-Savart law, have no cross product. In general, this makes these integrals easier to evaluate. As is seen by the following example, however, problems may arise when dealing with idealized sources, for example, line currents which are infinitely long, etc..

The previous (approximately) 2 pages should be handwritten notes on the example using a vector potential.

Ampère's Law implies Biot-Savart

We have Ampère's law in point form as

$$\vec{\nabla} \times \vec{H} = \vec{J} \qquad \text{(I)}$$

and using the vector potential of (1.34) along with equations (1.25) and (1.29)

$$\vec{H} = \vec{\nabla} \times \int_{v'} \frac{\vec{J}(\vec{r'})dv'}{4\pi\,|\vec{r} - \vec{r'}|}. \quad \text{(II)}$$

Note: $\vec{A}(\vec{r}) \equiv \vec{A}(x, y, z)$ so, in cartesian coordinates, the curl in (II) is with respect to the unprimed coordinates (x, y, z).

From Appendix 3.A we have for any vector field $\vec{G}$ and scalar C

$$\vec{\nabla} \times (C\vec{G}) = \vec{\nabla}C \times \vec{G} + C\vec{\nabla} \times \vec{G}. \quad \text{(III)}$$

Identifying $C \equiv |\vec{r} - \vec{r'}|$ and $\vec{G} \equiv \vec{J}(\vec{r'})$, and interchanging the order of integration and differentiation (which is allowable for well-behaved integrands) in (II), (III) becomes

$$\text{(IV)}$$

because $\vec{J}(\vec{r'})$ is NOT a function of (x, y, z) but of (x', y', z') (which means $\vec{\nabla}_{x,y,z} \times \vec{J}(\vec{r'}) = 0$). It is easily verified (DO THIS) that

where

$$\frac{1}{|\vec{r} - \vec{r'}|} =$$

so that (IV) becomes

$$\vec{\nabla} \times \left\{ \frac{1}{|\vec{r} - \vec{r'}|} \vec{J}(\vec{r'}) \right\} =$$

$$=$$

Substituting this last expression into (II) gives

$$\vec{H} =$$

which is exactly the Biot-Savart law of equation (1.20). Working this procedure in reverse would produce Ampère's law from the Biot-Savart law.

Justification for Setting $\vec{\nabla} \cdot \vec{A} = 0$ – for Non-time-varying Currents

Preliminaries:

Recall from an earlier course the definition of the convolution of functions, $f(t)$ and $g(t)$, for example:

$$\begin{aligned} f(t) \overset{t}{*} g(t) &= \int f(t')g(t-t')dt' \\ &= \int f(t-t')g(t')dt' \\ &= h(t), \quad \text{say.} \end{aligned}$$

Taking derivatives we find

$$\begin{aligned} \frac{d}{dt}\left\{f(t) \overset{t}{*} g(t)\right\} &= \frac{d}{dt}\int f(t')g(t-t')dt' \\ &= \int f(t')\frac{d}{dt}g(t-t')dt' \\ &= f(t) \overset{t}{*} \frac{dg(t)}{dt} \end{aligned}$$

OR

$$\begin{aligned} \frac{d}{dt}\left\{f(t) \overset{t}{*} g(t)\right\} &= \frac{d}{dt}\int f(t-t')g(t')dt' \\ &= \int \frac{d}{dt}f(t-t')g(t-t')dt' \\ &= \frac{df(t)}{dt} \overset{t}{*} g(t) \end{aligned}$$

So, for convolutions,

$$\frac{d}{dt}(f * g) = \frac{df}{dt} * g = f * \frac{dg}{dt} = \frac{dh}{dt}.$$

Finally we note that

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

Using these preliminary considerations we return to the main problem of showing that $\vec{\nabla} \cdot \vec{A} = 0$ for non-time-varying currents.

With a view to using the above, we write (34)

explicitly as

where $\overset{3d}{*}$ represents a 3-dimenensional *spatial* convolution. Now,

$$
\begin{aligned}
A_x(x, y, z) &= \\
A_y(x, y, z) &= \\
A_z(x, y, z) &=
\end{aligned}
$$

On the basis of the preliminaries, we may write the derivatives of the last expressions as

So that

But for non-time-varying currents, the equation of continuity gives

Specification of the vector potential in this way is called *setting the gauge* and $\vec{\nabla} \cdot \vec{A} = 0$, in particular, is the the so-called *Coulomb gauge*. Remember, for a complete (unique) specification, both $\vec{\nabla} \times \vec{A}$ and $\vec{\nabla} \cdot \vec{A}$ must be established.

The Lorentz Force Equation and Related Concepts

Force on a Moving Charge:

Recall that a charge, Q, in an electric field $\vec{E}$, experiences a force, $\vec{F}_e$, given as

$$\vec{F}_e = Q\vec{E} \tag{1.37}$$

Note that $\vec{F}_e$ is in the same direction as $\vec{E}$ and can therefore cause a linear acceleration of the charge – i.e. the force can change the charge's velocity, thus imparting or removing kinetic energy. Another way of saying this is that the force is capable of doing work on the charge. Recall that the acceleration vector, $\vec{a}$, by Newton's second law of motion ($\vec{F} = m\vec{a}$, where m is mass) is in the same direction as the force.

Next, a charge moving with a velocity $\vec{v}$ in a magnetic field of flux density $\vec{B}$ experiences a force given by

$$\vec{F}_m = Q(\vec{v} \times \vec{B}). \tag{1.38}$$

We note from the cross product that this force is perpendicular to both the field vector and the velocity vector. Of course, a force of this nature may cause a change in direction of $\vec{v}$ but not in its magnitude (the analogy in mechanics is the centripetal force on a mass moving with *constant* speed in a circular path). Thus, since $\vec{F}$ causes no change in $|\vec{v}|$, it's neither capable of increasing nor decreasing the energy of the charge involved.

Equations (1.37) and (1.38) hold for both steady and time-varying fields.

Combining the electric and magnetic field effects, we get fro equations (1.37) and (1.38)

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B}). \tag{1.39}$$

This is the famous LORENTZ FORCE EQUATION. It applies to any charge motion in electric and magnetic fields (particle accelerators, magnetrons, CRT's etc.).

Example: Determine the velocity relations and trajectory (path of motion) for a positive charge q of mass m moving in a magnetic field specified by $\vec{B} = B_0\hat{z}$ if at time $t = 0$, the charge is located at the origin and has a velocity given by $\vec{v} = v_0\hat{y}$.

Force on Current Elements (Line, Surface and Volume):

Charges and Line Charges:

If we agree to drop the m subscript from equation (1.38), but remember from the context that $Q(\vec{v} \times \vec{B}$ is force due to the charge moving in a magnetic field, then a differential force $d\vec{F}$ on a differential charge dQ is given by

$$d\vec{F} = dQ\vec{v} \times \vec{B} \tag{1.40}$$

Recalling that for a constant line current I, $dQ = Idt$ and that the velocity is $\vec{v} = d\vec{L}/dt$, it is easy to see that equation (1.40) may be written for a differential line current as

$$d\vec{F} = Id\vec{L} \times \vec{B} \tag{1.41}$$

Illustration:

Integrating over a closed path (circuit) and assuming the current to be constant, (1.41) leads to

$$\vec{F} = \oint_C Id\vec{L} \times \vec{B} = -I \oint_C \vec{B} \times d\vec{L} \tag{1.42}$$

SPECIAL CASE 1: Furthermore, if the field is uniform so that $\vec{B}$ may also be removed from the integral in (1.42) it becomes obvious that the *total* force on the circuit is zero:

This does NOT mean that the torque is zero! (Think about force couples encountered in mechanics).

SPECIAL CASE 2: Suppose next that we consider applying (1.42) to a portion of a straight conductor in a constant $\vec{B}$ field – the path is no longer closed. We have

$$\vec{F} = I \int_C d\vec{L} \times \vec{B} = -I\vec{B} \times \int_C d\vec{L} = -I\vec{B} \times \vec{L}$$

or

$$\vec{F} = I\vec{L} \times \vec{B}.$$

Thus, the magnitude of the force is given by the familiar expression

$$\left|\vec{F}\right| = BIL\sin\theta \tag{1.43}$$

where θ is the angle between $\vec{L}$ and the field vector. Of course, if this is to be applied to an extended piece of current-carrying conductor, the conductor must be straight and the field must be uniform.

<u>Forces Experienced by Volume and Surface Current Densities:</u>

In the prerequisite to this course, the (volume) current density was defined as

$$\vec{J} = \rho_v \vec{v}$$

and noting that $dQ = \rho_v dv$ we have from (1.40) that $d\vec{F} = \rho_v dv \vec{v} \times \vec{B}$, which implies

$$d\vec{F} = \vec{J} \times \vec{B} dv \tag{1.44}$$

Thus, the total force on a finite volume is given by

$$\vec{F} = \int_{\text{vol}} \vec{J} \times \vec{B} dv \tag{1.45}$$

It may be similarly argued for surface current density, $\vec{K}$ that

$$d\vec{F} = \vec{K} \times \vec{B} dS \tag{1.46}$$

Thus, the total force for a finite surface is given by

$$\vec{F} = \int_S \vec{K} \times \vec{B} dS \tag{1.47}$$

Read the paragraphs on the Hall voltage in Section 9.2 of the text and know how to explain the effect in terms of equation (1.40).

Torque on a Closed Circuit

While the force on a closed (dc) circuit in a uniform $\vec{B}$ field is zero, *the moment of force* or *torque* is not generally zero.

Definition: The torque, $\vec{T}$, or moment of force is defined with respect to the accompanying diagram as follows:

$$\vec{T} = \vec{R} \times \vec{F} \tag{1.48}$$

(with reference to the origin in this illustration).

Note that $\vec{T}$ is perpendicular to the plane containing $\vec{R}$ and $\vec{F}$ and points in the direction determined by the usual right-hand rule for cross products. The magnitude of $\vec{T}$ is then

$$|\vec{T}| = |\vec{R}||\vec{F}| \sin\theta \tag{1.49}$$

(i.e. $T = RF\sin\theta$).

Next consider a force *couple*, $\vec{F}_1$ and $\vec{F}_2$ so that $\vec{F}_1 + \vec{F}_2 = 0$ or $\vec{F}_1 = -\vec{F}_2$. This couple is applied to some "object" in space as shown:

Illustration:

$(\vec{R}_2 - \vec{R}_1)$ is the "object" between points P_2 and P_1 of the force applications.

The net torque about an axis through the origin ($\perp$ to page) is given by which indicates that $\vec{T}$ is independent of the choice of origin for the position vectors, $\vec{R}_1$ and $\vec{R}_2$. The individual moment arms have disappeared and the vector $(\vec{R}_1 - \vec{R}_2)$ between the points of application has emerged.

Magnetic Torque on a Small Closed Loop (of Differential Area):

Consider, without loss of generality, a differential current loop in the $x - y$ plane centered at the origin O as shown.

Illustration:

If the dimension of the circuit is small and $\vec{B}$ is continuous (but not necessarily uniform), then over the small loop $\vec{B}$ may be considered as some constant equal to the value of the field at the origin (i.e., $\vec{B} = \vec{B}_0$) – this is the zero-order effect.

For side 1,

Therefore, for side 1, the differential torque, $d\vec{T}_1$, about an axis through O is given by

Pages 27 and 28 should be done on looseleaf and end with equation (1.52).

SPECIAL CASE: Constant $\vec{B}$ and I

If the applied $\vec{B}$ field is uniform and the current is uniform in any *planar* loop, we may integrate equation (1.52) to obtain

Using the magnetic dipole moment,

$$\vec{m} = IS\hat{n} \ ,$$

with S being the total surface area of any planar loop enclosed by a current, I, the torque becomes

$$\vec{T} = \vec{m} \times \vec{B} \ . \qquad (1.53)$$

<u>Illustration:</u>

We see that $\vec{T}$ tends to rotate the loop such that $\vec{m}$ $(= \hat{n}IS)$ and the induced field $\vec{B}_I$ tend toward the $\vec{B}$ direction. That is, the *induced* field tends to line up with the *impressed* or applied field so as to maximize the total $\vec{B}$-field. We note that in the magnitude sense,

(1.54)

Therefore,

$$\text{if } \hat{n} \perp \vec{B}\,, \qquad T \to T_{\text{max}}$$
$$\text{if } \hat{n} \,||\, \vec{B}\,, \qquad T \to 0$$

The foregoing serves as a basis for modeling the nature of magnetic materials (and classifying them as *diamagnetic, paramagnetic, ferromagnetic, antiferromagnetic, ferrimagnetic* or *superparamagnetic*). Please read the text, Section 9.5, pp 288-292 for a description of the classes of magnetic materials.

Magnetization and Permeability

The motion of the *bound* charges (i.e. NOT *free* charge) inside a material due to electrons orbiting and spinning and nucleii spinning may have associated with it some sort of *net* bound current, I_b. We had earlier the differential magnetic dipole moment, $d\vec{m} = Id\vec{S}$ where $d\vec{S}$ is the vector differential area enclosed by the current, I. If we choose to rename the magnetic dipole moment as $\vec{m}$ (rather than $d\vec{m}$) for the differential area,

$$\vec{m} = I_b d\vec{S} \qquad (1.55)$$

due to the bound current. Consider a sample of material containing n magnetic dipole moments per unit volume (so that in a volume Δv, there are $(n\Delta v)$ dipoles).

The total dipole moment (a vector sum) may thus be written as

(1.56)

This could be zero, and often is if the m_i's are randomly distributed. However, it is possible that the total dipole moment is non-zero and this is especially true if the material is subjected to a $\vec{B}$-field or has a history of being so subjected.

By definition, the *magnetization*, $\vec{M}$, is defined as

$$\vec{M} = \lim_{\Delta v \to 0} \frac{\vec{m}_{\text{total}}}{\Delta v} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} \vec{m}_i \tag{1.57}$$

Thus, $\vec{M}$ has units of A/m which is the same as the units of $\vec{H}$. Consequently, $\vec{M}$ may be regarded as a kind of $\vec{H}$ field arising from the alignment of the $\vec{m}_i$'s within the material when an *external* $\vec{H}$ (or $\vec{B}$) field is applied. Thus, if this alignment occurs, the total $\vec{B}$ field appears larger, in general, than it would be if $\vec{M}$ were not present. [Remember, $\vec{H}$ is due to the "free" current; $\vec{M}$ is due to the "bound" current].

Now,

and

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) \tag{1.58}$$

where $\vec{B}$ is the total magnetic flux density, $\vec{H}$ arises from "external" or free current and $\vec{M}$ accounts for all of the atomic magnetic effects. This may be further emphasize by writing

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \; . \tag{1.59}$$

Special Case: Linear Isotropic Material

In general, magnetic materials are not linear – for example, a change in the external field impinging a magnetic material will not, in general, cause a proportional change in the magnetic field within the material. However, for small flux densities (low currents) it is possible that a material may be considered to be linear: In particular, we assume that the magnetization $\vec{M}$ depends in a linear way on the applied $\vec{H}$ field and may thus be modeled as

$$\vec{M} = \chi_m \vec{H} \tag{1.60}$$

where χ_m is dimensionless and is referred to as the *magnetic susceptibility* of the material. Using (1.60), equation (1.58) may be cast as

$$\begin{aligned} \vec{B} &= \mu_0(\vec{H} + \chi_m \vec{H}) \\ \vec{B} &= \mu_0(1 + \chi_m)\vec{H} \end{aligned}$$

or

$$\vec{B} = \mu \vec{H} \tag{1.61}$$

where μ $(= \mu_0(1 + \chi_m))$is the permeability of the medium under consideration. We define a dimesionless constant called the *relative permeability*, μ_R, of a medium as

$$\mu_R = (1 + \chi_m) = \frac{\mu}{\mu_0} \tag{1.62}$$

where, of course, μ_0 is the permeability of free space.

[NOTE: In anisotropic magnetic media, μ is, in geneal, a 3×3 matrix.]

<u>Example:</u> Problem 19(a,b,c), page 318 of text.

The Boundary Conditions for Magnetic Fields

Just as was done for the static electric field, we now wish to consider the situation in which magnetic fields exist at a boundary between two media of differing magnetic properties. Boundaries represent mathematical discontinuities so that, at such interfaces, the derivatives encountered in Maxwell's equations do not strictly exist. Thus, in addition to the equations which we already have, we should develop a set of "boundary conditions" so that the former equations may be supplemented to handle e-m fields at the media interfaces.

Consider the figure below, where the interface is basically defined by (homogeneous) regions of differing permeabilities.

Referring to the CLOSED cylindrical surface on the left and using Gauss' law for the magnetic field we write $\oint_S \vec{B} \cdot d\vec{S} = 0$ where for the top base of the region, $\Delta\vec{S} = \Delta S \hat{n}_1$ and for the bottom base, $\Delta\vec{S} = \Delta S \hat{n}_2$. Thus, Gauss' law may be written as

(1.63)

In the limit as the sides of the cylindrical surface tend toward 0 (i.e. as the boundary is approached from "top" and "bottom" – see figure on the right above) it is clear that the unit normals to the surface may be written as

$$\hat{n}_1 \to \hat{n}$$

$$\hat{n}_2 \to -\hat{n}$$

and (side contribution) $\rightarrow 0$. Equation (63) becomes

$$\vec{B}_1 \cdot \hat{n} - \vec{B}_2 \cdot \hat{n} = 0$$

or using the subscript N to indicate "normal" components we have

$$B_{N_1} = B_{N_2} \tag{1.64}$$

This implies that the normal component of the $\vec{B}$ field is *continuous* across the boundary. Given that magnetic field constitutive relationship, it immediately follows that

$$H_{N_2} = \frac{\mu_1}{\mu_2} H_{N_1} \tag{1.65}$$

and it is seen that the normal component of the magnetic field intensity is discontinuous across the boundary.

Next, to investigate the tangential components of the $\vec{H}$ fields at the boundary, consider the figure below:

Let's apply Ampère's law to the closed path. If we allow for the possibility that a uniform surface current density, $\vec{K}$, may exist on the boundary and that the component of this density normal to the closed path is K, then

In the limit, as the ends tend toward 0,

or

$$H_{t_1} - H_{t_2} = K \tag{1.66}$$

Of course, if no surface current density exists on the boundary, we have the following very important simplification:

$$H_{t_1} - H_{t_2} = 0 \tag{1.67}$$

In terms of the unit normal, $\hat{n}$, to the surface, it is easy to see that equations (1.66) and (1.67) may be written as

and

respectively.

Equation (1.67) implies that, if tehre are no surface currents, the tangential $\vec{H}$ fields are continuous at the boundary. *This has very important consequences!* – stay tuned.