Unit 3

The Uniform Plane Wave and Related Topics

3.1 The Helmoltz Equation and its "Solution"

In this unit, we shall seek the physical significance of the Maxwell equations, summarized at the end of Unit 2, in a more general manner. That is, we ask the question "What is the physical nature of the solution of this set of first-order, coupled partial differential equations (p.d.e)?". In so doing, let us consider the equations in point form, eliminate one of the variables (say $\underset{\sim}{\vec{H}}$), and "solve" the resulting equation for the $\underset{\sim}{\vec{E}}$-field. From the summarized equations, it is clear that we have for free space

1.

2.

3.

4.

To eliminate $\underset{\sim}{\vec{H}}$ from the above and obtain a single p.d.e. in one "unknown", let's take the curl again on both sides of equation (3.1) and interchange the curl and time derivative:

From (3.2) this gives

Now, from the text, Appendix A.3, we have for any vector $\vec{A}$

Therefore,

$$\vec{\nabla}(\vec{\nabla}\cdot\underset{\sim}{\vec{E}})-\vec{\nabla}^2\underset{\sim}{\vec{E}}=-\mu_0\epsilon_0\frac{\partial^2\underset{\sim}{\vec{E}}}{\partial t^2} \qquad \text{(A)}$$

and since for free space equation (3.3) holds, equation (A) becomes

$$\vec{\nabla}^2\underset{\sim}{\vec{E}}-\mu_0\epsilon_0\frac{\partial^2\underset{\sim}{\vec{E}}}{\partial t^2}=0 \tag{3.5}$$

This is the (vector) Helmoltz equation. Had we carried out a similar procedure to eliminate $\underset{\sim}{\vec{E}}$ starting with equation (3.2), we would have obtained for the $\underset{\sim}{\vec{H}}$-field

$$\vec{\nabla}^2\underset{\sim}{\vec{H}}-\mu_0\epsilon_0\frac{\partial^2\underset{\sim}{\vec{H}}}{\partial t^2}=0 \tag{3.6}$$

Equations (3.5) and (3.6) are "wave equations" which is, as we shall see, an appropriate name. (Their general form is that of hyperbolic p.d.e's – see a text on p.d.e.'s.)

Let us seek a simple solution for $\underset{\sim}{\vec{E}}$ in equation (3.5) by assuming that the electric field intensity satisfying that equation has only an x-component which depends only on z spatially; that is

$$\underset{\sim}{\vec{E}}=\underset{\sim}{E}_x(z,t)\,\hat{x}\ .$$

Clearly, then, $\vec{\nabla}^2\underset{\sim}{\vec{E}}=\frac{\partial^2\underset{\sim}{\vec{E}}}{\partial x^2}+\frac{\partial^2\underset{\sim}{\vec{E}}}{\partial y^2}+\frac{\partial^2\underset{\sim}{\vec{E}}}{\partial z^2}$ has only one surviving term given by

$$\vec{\nabla}^2\underset{\sim}{\vec{E}}=\frac{\partial^2\underset{\sim}{E}_x}{\partial z^2}\ \hat{x} \tag{3.7}$$

Also, the time derivative is given as

$$\frac{\partial^2\underset{\sim}{\vec{E}}}{\partial t^2}=\frac{\partial^2}{\partial t^2}\left[\underset{\sim}{E}_x(z,t)\hat{x}\right]=\frac{\partial^2\underset{\sim}{E}_x}{\partial t^2}\ \hat{x} \tag{3.8}$$

Then, equation (3.5) becomes from (3.7) and (3.8)

$$\left[\frac{\partial^2\underset{\sim}{E}_x}{\partial z^2}-\mu_0\epsilon_0\frac{\partial^2\underset{\sim}{E}_x}{\partial t^2}\right]\ \hat{x}=0$$

which implies

$$\frac{\partial^2\underset{\sim}{E}_x}{\partial z^2}-\mu_0\epsilon_0\frac{\partial^2\underset{\sim}{E}_x}{\partial t^2}=0\ . \tag{3.9}$$

This is a one-dimensional "wave" equation in free space. We will next verify that

$$\underset{\sim}{E}_x(z,t)=E_0\cos\left[\omega\left(t-\sqrt{\mu_0\epsilon_0}z\right)\right] \tag{3.10}$$

is at least one solution of equation (3.9). Now, and, obviously, (3.10) satisfies (3.9) [and it also satisfies Maxwell's equations given the stipulation that $\vec{\nabla} \cdot \underset{\sim}{\vec{E}} = 0$ for the region being considered – i.e. free space].

Now, the corresponding $\underset{\sim}{\vec{H}}$-field may be determined from $\vec{\nabla} \times \underset{\sim}{\vec{E}} = -\frac{\partial \underset{\sim}{\vec{B}}}{\partial t}$ as follows:

Finally, integrating gives

or

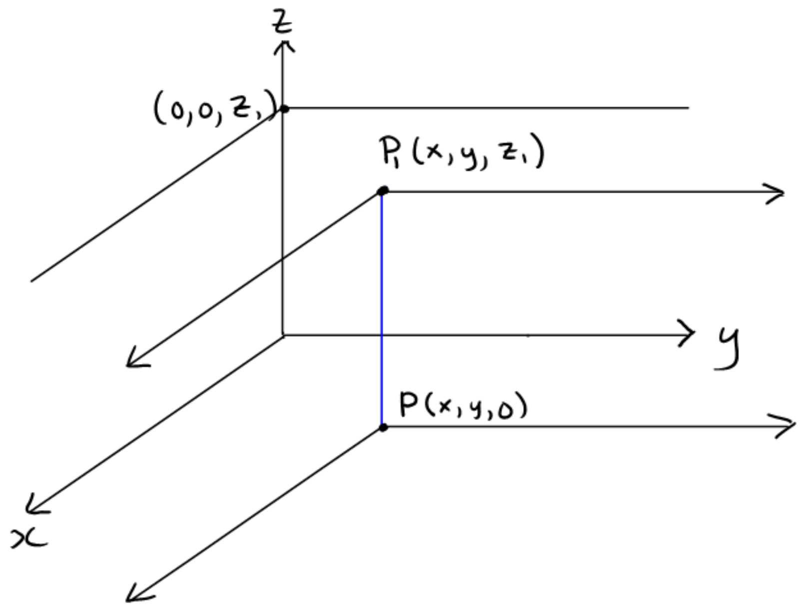
$$\underset{\sim}{\vec{H}}(z,t) = \frac{E_0}{\sqrt{\mu_0/\epsilon_0}} \cos\left[\omega\left(t - \sqrt{\mu_0 \epsilon z}\right)\right] \hat{y} \tag{3.11}$$

Comparing with equation (3.10),

$$\underset{\sim}{\vec{H}}(z,t) = \frac{\underset{\sim}{E}_x(z,t)}{\sqrt{\mu_0/\epsilon_0}} \hat{y} \tag{3.12}$$

The reason for writing the coefficient on the cosine as shown will be seen shortly.

From the above analysis, we see that a time-varying E-field implies a time-varying H-field. Let's now interpret the results.



At a given instant (i.e. fixed t), all points, P, in the x–y plane have a constant phase. Similarly all points, P_1, in the $z = z_1$ plane have a (different) constant phase. That is, all observers in a particular plane parallel to the x–y plane see the same fields and these time-varying $\underset{\sim}{\vec{E}}$ and $\underset{\sim}{\vec{H}}$ fields are phase delayed as one progresses in the z-direction. Thus, since all observers in a plane see the same fields, equations (3.10) and (3.11) combined are referred to as *plane waves.* Furthermore a plane wave with uniform amplitudes over its constant phase planes is called a *uniform* plane wave.

Temporal Period, T:

The time period, T, of the sinusoidal fields is defined as that time such that

$$\omega T = 2\pi \ \text{ or } \ 2\pi f T = 2\pi \ \Rightarrow \boxed{f = \frac{1}{T}}$$

where f is the frequency in hertz.

Wavelength, λ_0, and wavenumber, k_0:

We may regard the $\omega\sqrt{\mu_0\epsilon_0}$ coefficient on z as a "spatial frequency", k_0, commonly referred to as the free space *wavenumber*, and define the "spatial period" (i.e. *wavelength*), λ_0, as

$$\omega\sqrt{\mu_0\epsilon_0}\lambda_0 = 2\pi \ \text{ or } \ \lambda_0 = \frac{2\pi}{\omega\sqrt{\mu_0\epsilon_0}} = \frac{2\pi}{2\pi f\sqrt{\mu_0\epsilon_0}}$$

or

$$\boxed{\lambda_0 = \frac{2\pi}{k_0}} \tag{3.13}$$

All observers in planes separated by integer multiples of λ_0 see the same fields (i.e. no phase difference).

We note, too, that a *wave vector* $\vec{k_0}$ may be defined as

$$\boxed{\vec{k_0} = k_0\hat{k}}$$

where $\hat{k}$ is a unit vector in the direction of travel (see below). In the illustration, we are using, $\hat{k} = \hat{z}$. Many texts used k instead of k as the wavenumber symbol. In general, the subscript 0 will be used only to represent free-space values.

Phase velocity

Suppose, next, that the observer travels along with the e-m field so that a constant field is observed. This clearly requires that

$$\omega t - \omega\sqrt{\mu_0\epsilon_0}z = \text{ constant}$$

which on taking the time derivative yields

$$\omega - \omega\sqrt{\mu_0\epsilon_0}\frac{\partial z}{\partial t} = 0 \ \ .$$

Defining the derivative to be the *phase velocity*, v_p,

$$\boxed{v_p = \frac{\partial z}{\partial t} = \frac{\omega}{k_0} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c \approx 3 \times 10^8 \text{ m/s}} \tag{3.14}$$

where c is the speed of e-m waves (i.e. "light") in free space (vacuum).

Intrinsic Impedance, η_0, of Free Space:

It may be readily observed that the quantity $\sqrt{\frac{\mu_0}{\epsilon_0}}$ has ohms as its unit, and this quantity is referred to as the *intrinsic impedance* of free space:

$$\boxed{\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 120\pi\ \Omega}$$

Other Possible Solutions

We have claimed that equations (3.10) and (3.11) form one free-space solution to Maxwell's equations. They are certainly NOT the only solutions. Consider the following:

3.2 Time-harmonic Analysis of Maxwell's Equations

While, as seen above, there are many possible free-space solutions to Maxwell's equations. Let us concentrate on the so-called *time-harmonic* forms – i.e. sinusoidal form – in which case, phasor notation becomes useful:

Recall that a function A of the form

$$A(t) = A_0 \cos(\omega t + \phi)$$

may be written as

$$A(t) = \mathcal{R}e\left\{Ae^{j\omega t}\right\} \tag{3.15}$$

where $A = A_0 e^{j\phi}$ is called the phasor form of $A(t)$. It is easy to see that a time-derivative of $A(t)$ has a (complex) phasor form of $j\omega A$. Here, ω is radian frequency and ϕ is any non-time-dependent phase angle.

On this basis, the Maxwell equations may be written for time-harmonic fields in phasor form as:

We have used the subscript "s" to indicate phasor notation.

Proceeding as before, we take the curl of (3.16) to get

Using the usual vector identity on the left and substituting from equation (3.17) for the curl on the right, we get which for free space with $\vec{\nabla} \cdot \vec{E}_s = 0$ gives

$$\boxed{\vec{\nabla}^2 \vec{E}_s + \omega^2 \mu_0 \epsilon_0 \vec{E}_s = 0 = \vec{\nabla}^2 \vec{E}_s + k_0^2 \vec{E}_s} \tag{3.22}$$

where $k_0^2 = \omega_0^2 \mu_0 \epsilon_0$.

Again, we seek a simple solution $\vec{E}_s$ which has only an x component and a z dependence; i.e.,

$$\vec{E}_s = E_{x_s}(z)\,\hat{x}$$

Notice the removal of the time dependence (we are using phasors). Considering the pieces of equation (3.22) we get

Therefore, from (3.22)

This equation has a general solution (see your differential equation notes from another course)

where E_{x0}^- and E_{x0}^+ are constants. Since we are dealing with phasors, these constants could be complex. The two pieces of the solution are themselves independent solutions. The $-$ and $+$ in the constants will be interpreted to indicate the directions of travel. You should be convinced of this soon. Let us first consider the second solution, $E_{x0}^+ e^{-jk_0 z}$:

We may find $\vec{H}_s$ from (3.16):

Using the result for the phasor electric field as above, we get

and using (3.15),

the time domain form becomes

$$\underset{\sim}{\vec{H}} = \frac{E^+_{x0}}{\sqrt{\mu_0/\epsilon_0}} \cos(\omega t - k_0 z)\,\hat{y}$$

with

Suppose, next, that we no longer have free-space values for μ and ϵ, but that the other properties still hold (namely, $\rho_v = 0$ so that $\vec{\nabla} \cdot \vec{E} = 0$). The results for the above parameters would clearly become

3.3 Power Flow and the Poynting Vector

Let's again consider Maxwell's equations in point form for non-free space and (for simplicity) linear media so that μ and ϵ are ordinary scalars:

$$\vec{\nabla} \times \underset{\sim}{\vec{E}} = -\frac{\partial \underset{\sim}{\vec{B}}}{\partial t} \quad (3.23) \qquad \vec{\nabla} \times \underset{\sim}{\vec{H}} = \underset{\sim}{\vec{J}} + \frac{\partial \underset{\sim}{\vec{D}}}{\partial t} \quad (3.24)$$

$$\vec{\nabla} \cdot \underset{\sim}{\vec{D}} = \underset{\sim}{\rho_v} \quad (3.25) \qquad \vec{\nabla} \cdot \underset{\sim}{\vec{B}} = 0 \quad (3.26)$$

We would like to get some idea of the power flow associated with the e-m waves described by the these equations. We'll need some of our usual vector identity "techniques".

<u>STEP 1:</u> Take the dot product of both sides of equation (3.24) with $\vec{E}$:

$$\underset{\sim}{\vec{E}} \cdot (\vec{\nabla} \times \underset{\sim}{\vec{H}}) = \underset{\sim}{\vec{E}} \cdot \underset{\sim}{\vec{J}} + \underset{\sim}{\vec{E}} \cdot \frac{\partial \underset{\sim}{\vec{D}}}{\partial t} \tag{3.27}$$

<u>STEP 2:</u> Take the dot product of both sides of (3.23) with $\underset{\sim}{\vec{H}}$:

$$\underset{\sim}{\vec{H}} \cdot (\vec{\nabla} \times \underset{\sim}{\vec{E}}) = -\underset{\sim}{\vec{H}} \cdot \frac{\partial \underset{\sim}{\vec{B}}}{\partial t} \tag{3.28}$$

<u>STEP 3:</u>Subtract the corresponding members of (3.28) from (3.27) to get

$$\underset{\sim}{\vec{E}} \cdot (\vec{\nabla} \times \underset{\sim}{\vec{H}}) - \underset{\sim}{\vec{H}} \cdot (\vec{\nabla} \times \underset{\sim}{\vec{E}}) = \underset{\sim}{\vec{E}} \cdot \underset{\sim}{\vec{J}} + \underset{\sim}{\vec{E}} \cdot \frac{\partial \underset{\sim}{\vec{D}}}{\partial t} + \underset{\sim}{\vec{H}} \cdot \frac{\partial \underset{\sim}{\vec{B}}}{\partial t} \tag{3.29}$$

and note from Appendix A.3 of the text that for two vectors, $\vec{A}$ and $\vec{B}$, for example

Comparing this with equation (3.29) shows

$$-\vec{\nabla} \cdot (\underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{H}}) = \underset{\sim}{\vec{E}} \cdot \underset{\sim}{\vec{J}} + \underset{\sim}{\vec{E}} \cdot \frac{\partial \underset{\sim}{\vec{D}}}{\partial t} + \underset{\sim}{\vec{H}} \cdot \frac{\partial \underset{\sim}{\vec{B}}}{\partial t} \tag{3.30}$$

<u>STEP 4:</u> Under our assumption of linear media, $\underset{\sim}{\vec{D}} = \epsilon \underset{\sim}{\vec{E}}$ and $\underset{\sim}{\vec{B}} = \mu \underset{\sim}{\vec{H}}$ so that (3.30) may be written as

$$-\vec{\nabla} \cdot (\underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{H}}) = \underset{\sim}{\vec{E}} \cdot \underset{\sim}{\vec{J}} + \epsilon \underset{\sim}{\vec{E}} \cdot \frac{\partial \underset{\sim}{\vec{E}}}{\partial t} + \mu \underset{\sim}{\vec{H}} \cdot \frac{\partial \underset{\sim}{\vec{H}}}{\partial t} \tag{3.31}$$

We make the following observations on the time derivatives:

This allows equation (3.31) to be written as

$$-\vec{\nabla} \cdot (\underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{H}}) = \underset{\sim}{\vec{E}} \cdot \underset{\sim}{\vec{J}} + \frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon \underset{\sim}{E}^2\right] + \frac{\partial}{\partial t}\left[\frac{1}{2}\mu \underset{\sim}{H}^2\right] \tag{3.32}$$

Notice that each member of this equation is a scalar quantity – i.e. this is a *scalar* equation. We now want to address the question "What does each term signify?". To convince ourselves of the answer, consider the units:

Thus, observing the units and observing the divergence and derivatives in equation (3.32), we see that this is the "point form" of a *power balance* equation. Let's try to get an overall picture of this power balance by integrating (3.32) over a volume V which is *enclosed* by a surface S. Writing this explicitly we get

Recall the divergence theorem (Gauss' integral theorem) for any vector field $\vec{A}$,

$$\int_V \vec{\nabla} \cdot \vec{A} dv = \oint_S \vec{A} \cdot d\vec{S} \ .$$

Applying this to the left hand side of the ;last equation gives

$$-\oint_S (\underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{H}}) \cdot d\vec{S} = \int_V \underset{\sim}{\vec{E}} \cdot \underset{\sim}{\vec{J}} dv + \frac{\partial}{\partial t}\int_V \left[\frac{1}{2}\epsilon \underset{\sim}{E}^2 + \frac{1}{2}\mu \underset{\sim}{H}^2\right] dv \tag{3.33}$$

From our discussion above, it is clear that each of these integrals has units of watts (i.e. power units).

TERM by TERM CONSIDERATION of (3.33):

$$\int_V \underset{\sim}{\vec{E}} \cdot \underset{\sim}{\vec{J}} dv \quad :$$

$$\frac{\partial}{\partial t} \int_V \left[\frac{1}{2} \epsilon \underset{\sim}{E}^2 + \frac{1}{2} \mu \underset{\sim}{H}^2 \right] dv \quad :$$

$$- \oint_S (\underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{H}}) \cdot d\vec{S} \quad :$$

Remember, $d\vec{S}$ is the outward-pointing normal from the surface surrounding V. Therefore $- \oint_S (\underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{H}}) \cdot d\vec{S}$ by virtue of the "$-$" is power flowing *into* V through S. Therefore, equation (3.33) "reads":

power flowing into volume = [(power dissipated (if ohmic losses)) + (power stored)] within volume.

DEFINITION of the POYNTING VECTOR:

The product $\underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{H}}$, which has units of watts/metre^2 is called the *Poynting vector*:

$$\underset{\sim}{\vec{P}} = \underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{H}} \quad . \tag{3.34}$$

This is a quantity whose magnitude is a *power density* (W/m^2) and whose direction points in the direction of positive power flow. Notice that the Poynting vector is in a direction $\perp$ to the plane containing $\underset{\sim}{\vec{E}}$ and $\underset{\sim}{\vec{H}}$:

Power Flow: The power crossing some (not-necessarily closed) surface S due to an e-m source may clearly be given in terms of the Poynting vector:

The component of $\vec{\mathcal{P}}$ in the direction of the differential surface normal $d\vec{S}$ is the power density "passing through" the surface. The power P is then

$$\underset{\sim}{\mathcal{P}} = \int_S \underset{\sim}{\vec{\mathcal{P}}} \cdot d\vec{S} = \int_S (\underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{H}}) \cdot d\vec{S} \quad . \tag{3.35}$$

What does this imply about our plane waves? We had

(where we have not used free space values for η or k – this doesn't affect what follows). Clearly,

$$\underset{\sim}{\vec{\mathcal{P}}} = \underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{H}} = \frac{\left(E_{x0}^{+}\right)^2}{\eta} \cos^2(\omega t - kz)\, \hat{z} \text{ W/m}^2 \quad .$$

The time-averaged value of this Poynting vector is (by definition of "time-averaging")

(3.36)

Example: If we take a surface S which is everywhere parallel to the x-y plane (i.e. $z = z_0 =$ constant, then the average power flowing through S, given that $\vec{\mathcal{P}}$ has only a z component is simply

$$P_{\text{avg}} = \left|\vec{\mathcal{P}}_{\text{avg}}\right| S$$

in watts, which from (3.36) is

(3.37)

For surfaces not as simple as this, the integral

will have to be carried out. Note the analogy between (3.37) and the average power dissipated in a resistance R when a sinusoidal voltage of peak value V_p exists across it:

3.4 Plane Waves in Lossy Media

Let's again consider the first four equations of Section 3-3 under the assumption of time-harmonic fields so that we may write in phasor form

$$\vec{\nabla} \times \vec{E}_s = -j\omega\mu\vec{H}_s \quad (3.38) \qquad \vec{\nabla} \times \vec{H}_s = \vec{J}_s + j\omega\vec{D}_s \quad (3.39)$$

$$\vec{\nabla} \cdot \vec{D}_s = \rho_{v_s} \quad (3.40) \qquad \vec{\nabla} \cdot \vec{B}_s = 0 \quad (3.41)$$

Let the medium be characterized by

$$\vec{B}_s = \mu\vec{H}_s \quad (3.42) \quad \vec{D}_s = \epsilon\vec{E}_s \quad (3.43) \quad \vec{J}_s = \sigma\vec{E}_s \quad (3.44),$$

and we'll assume that there are no sources in the region under consideration. By virtue of the inclusion of Ohm's law (equation (3.44)) we are allowing for the fact that the medium may be dissipative or *lossy.* Equation (3.39), on using equations (3.43) and (3.44) may then be cast as

(3.45)

We note from (3.45) that the conduction current density $\vec{J}_{c_s}$ and the displacement current density $\vec{J}_{d_s}$ are 90° out of phase.

Carrying out exactly the same analysis as led to equation (3.22) of Section 3.2 (where we assumed $\sigma = 0$) and noting that because $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) \equiv 0$, $\vec{\nabla} \cdot \vec{E}_s = 0$ also, it is easily shown (SHOW THIS!!) that

$$\vec{\nabla}^2 \vec{E}_s - j\omega\mu(\sigma + j\omega\epsilon)\vec{E}_s = 0 \qquad (3.46)$$

We now define a new *propagation constant* (that replaces our earlier k) as

$$\gamma = \sqrt{(\sigma + j\omega\epsilon)j\omega\mu} \qquad (3.47)$$

This allows us to write (3.46) as

(3.48)

It may be easily seen from (3.47) and (3.48) that when the medium is *lossless* – i.e. $\sigma = 0$ – equation (3.48) reduces to (3.22) of Section 3.2.

Before considering a solution to (3.48), let's examine the new propagation constant in (3.47) a little more closely:

$$(3.49)$$

This *complex* propagation constant may be written as

$$\gamma = jk = \alpha + j\beta$$

where $\alpha\,,\,\beta \in \mathbb{R}$. Of course, the wavenumber k is now, in general, complex, and from (3.49) has the form

$$k =$$

If the only losses are due to the electric field, we may take μ as real while defining a *complex* permittivity ϵ_c by

$$\epsilon_c = \epsilon' - j\epsilon''$$

where

$$\epsilon' = \epsilon \text{ (the usual real permittivity) and } \epsilon'' = \sigma/\omega.$$

The complex wavenumber is then

$$k = \omega\sqrt{\mu\epsilon_c}\,.$$

It may be also observed from (3.45), that of the various quantities appearing in (3.49), the ratio $\dfrac{\sigma}{\omega\epsilon} = \dfrac{\epsilon''}{\epsilon'}$, called the *loss tangent*, is the ratio of the magnitude of the conduction current density to that of the displacement current density; i.e.

$$\frac{\sigma}{\omega\epsilon} = \frac{J_c}{J_d}\quad .$$

Finally, in order to specify particular α and β, the medium parameters (ϵ, μ, σ) and the wave frequency ω will have to be given. While we shall do this shortly in a relative way for some special cases, the general forms may be easily deduced from (3.49) as

$$\alpha = \mathcal{R}e\,\{\gamma\} = \mathcal{R}e\,\{jk\} = \tag{3.50}$$

and

$$\beta = \mathcal{I}m\left\{\gamma\right\} = \mathcal{I}m\left\{jk\right\} = \tag{3.51}$$

For the moment, let's again consider (3.48) and restrict our attention, as before, to fields having only one component as given by

$$\vec{E}_s = E_{x_s}(z)\hat{x}$$

Equation (3.48) may then be written in scalar form as

$$\frac{\partial^2 E_{x_s}(z)}{\partial z^2} - \gamma^2 E_{x_s}(z) = 0 \quad \text{or} \quad \frac{\partial^2 E_{x_s}(z)}{\partial z^2} + k^2 E_{x_s}(z) = 0 \; .$$

Again, from any d.e. text it may be verified that a solution to this equation, using constants E_{x0}^{+} and E_{x0}^{-} as before, is given by

(notice the two linearly independent solutions). As usual, we confine our attention to the solution

$$\vec{E}_{x_s}(z) = E_{x0}^{+}e^{-\gamma z}\hat{x} = E_{x0}^{+}e^{-\alpha z}e^{-j\beta z}\hat{x} \tag{3.52}$$

Repeating the techniques on (3.52) that were used in Section 3.2, it is easy to show (TRY IT) that

$$\vec{H}_s = \tag{3.53}$$

or

In keeping with earlier results, we now define the (complex) intrinsic impedance for the lossy medium as

$$\eta = \frac{j\omega\mu}{\gamma}$$

which from (3.49) becomes

$$\tag{3.54}$$

It is readily observed that if $\sigma = 0$, η is the same as before for "lossless" media.

Let's next take our solutions, equations (3.52) and (3.53), to the lossy medium wave equation back to the time domain:

$$\underset{\sim}{\vec{E}} \;=\; \mathcal{R}e\left\{\vec{E}_{x_s}(z)e^{j\omega t}\right\}$$

$$=$$

$$=$$

$$=$$

Taking E_{x0}^+ to be real, we have

$$\underset{\sim}{\vec{E}} = E_{x0}^+ e^{-\alpha z} \cos(\omega t - \beta z)\hat{x} \ . \tag{3.55}$$

Correspondingly, the magnetic field becomes

$$\underset{\sim}{\vec{H}} = \mathcal{R}e \left\{ \frac{\vec{E}_{x0}^+}{\eta} e^{-\alpha z} e^{j(\omega t - \beta z)} \right\} \hat{y} \ \ , \tag{3.56}$$

but now η is complex. Since this is the case we may write

$$\eta = \eta_m e^{j\theta_\eta}$$

where η_m is the *magnitude* of the intrinsic impedance and θ_η is its *phase*. Equation (3.56) then becomes

or

$$\underset{\sim}{\vec{H}} = \frac{E_{x0}^+}{\eta_m} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\hat{y} \ . \tag{3.57}$$

From the field expressions in (3.55) and (3.57) we make the following observations:

1. Both the $\underset{\sim}{\vec{E}}$ and $\underset{\sim}{\vec{H}}$ fields decay exponentially with distance travelled (in the z direction here) as seen by the $e^{-\alpha z}$ factor with $\alpha \in \mathbb{R}$ being called the *attenuation constant*. This constant is measured in units of nepers/metre (Np/m). Of course, the neper is dimensionless.

2. The $\underset{\sim}{\vec{E}}$ and $\underset{\sim}{\vec{H}}$ fields are out of phase by an amount θ_η, the phase of the intrinsic impedance η.

3. As before, the phase speed, v_p may be obtained as follows: Set

$$\omega t - \beta z = \text{constant}$$

   Take the derivative w.r.t. time to get the rate, v_p, at which the wave propagates in the z direction:

$$v_p = \frac{dz}{dt} = \frac{\omega}{\beta} \tag{3.58}$$

Remember, β is no longer necessarily $\omega\sqrt{\mu\epsilon}$ and v_p is no longer necessarily $1/\sqrt{\mu\epsilon}$ as was the case for non-dissipative media. However, it is still true that

$$\beta = \frac{2\pi}{\lambda}$$

4. The Poynting vector is now

   Recalling the identity

   and letting $A = (\omega t - \beta z)$ and $B = (\omega t - \beta z - \theta_\eta)$, we get

   (3.59)

   Using $<>$ to denote the time average

   (3.60)

   because the time-harmonic part of (3.59) integrates to zero. Note that $\cos\theta_\eta$ represents the "power factor" arising from the complex intrinsic impedance.

3.5 Propagation in Slightly Conducting and Highly Conducting Media

Let us again consider the general form of the propagation constant γ from the last section:

$$\gamma = j\omega\sqrt{\mu\epsilon}\sqrt{1 - j\frac{\sigma}{\omega\epsilon}} \tag{3.61}$$

and the quantity

$$\frac{\sigma}{\omega\epsilon} = \frac{J_c}{J_d} \quad . \tag{3.62}$$

We have noted that J_d leads the conduction current by 90°.

Clearly,

$$\frac{\sigma}{\omega\epsilon} = \tan\theta \tag{3.63}$$

and expression (3.63) is referred to as the *loss tangent* of the medium for the particular frequency of operation. We shall examine separately the cases where $\tan\theta \ll 1$ and where $\tan\theta \gg 1$.

CASE:1 Slightly Conducting Media: Good Dielectrics

When $\sigma/\omega\epsilon \ll 1$, the medium is slightly conducting for the frequency being considered i.e. the medium is not very lossy (good dielectric). Considering the square root in (3.61) for this case and recalling the binomial series

we write

$$\sqrt{1 - j\frac{\sigma}{\omega\epsilon}} = 1 - \frac{j}{2}\frac{\sigma}{\omega\epsilon} + \ldots + \text{smaller terms}$$

Then, from (3.61) the propagation constant may be approximated as

and, recalling that $\gamma = \alpha + j\beta$, we identify

$$\begin{aligned} \alpha &\approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \\ \beta &\approx \omega\sqrt{\mu\epsilon} \end{aligned}$$

Also, from the series approximation for γ, the intrinsic impedance from the equation preceeding (3.54) may be be developed as

Summary for Good Dielectrics: For future reference, we write the propagation constant and intrinsic impedance as

$$\gamma \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} + j\omega\sqrt{\mu\epsilon} \tag{3.64}$$

and

$$\eta \approx \sqrt{\frac{\mu}{\epsilon}}\left(1 + \frac{j\sigma}{2\omega\epsilon}\right) \ . \tag{3.65}$$

These may be used in equations (3.55) and (3.57) when $\sigma/\omega\epsilon \ll 1$.

CASE:2 Highly Conducting Media: Good Conductors

When $\sigma/\omega\epsilon \gg 1$, the medium is highly conducting for the frequency being considered. Thus, in conjunction with Case 1, we see that the loss tangent is an indicator of the conductive properties of the medium at specified frequencies.

This time the square root in (3.61) may be approximated as

$$\begin{aligned} \gamma &\approx j\omega\sqrt{\mu\epsilon} \cdot \sqrt{-j\frac{\sigma}{\omega\epsilon}} \\ &= \\ &= \\ &= \end{aligned}$$

Again noting $\gamma = \alpha + j\beta$, if $\sigma/\omega\epsilon \gg 1$, we observe

$$\alpha = \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}}$$

or writing $\omega = 2\pi f$, where f is frequency in hertz,

$$\alpha = \beta \approx \sqrt{\pi f \mu\sigma} \tag{3.66}$$

(Note: $\beta = 2\pi/\lambda$.)

Next, using the approximation of γ for the case at hand, we note that the intrinsic impedance may be written as

$$\eta = \frac{j\omega\mu}{\gamma} \approx$$

Therefore,

$$\eta = \tag{3.67}$$

Electric Field in Good Conductors – The Concept of Skin Depth

For the present case with $\alpha = \beta \approx \sqrt{\pi f \mu\sigma}$, equation (3.55) becomes

$$\tag{3.68}$$

Suppose we have a plane wave, whose E-field is given by (3.68), incident on a good conductor. Due to $\sigma \neq 0$, there will be ohmic loss (heat dissipation) and the wave amplitude decays exponentially according to

$$e^{-\alpha z} = e^{-\sqrt{\pi f \mu \sigma} z}$$

Definition: By definition, the skin depth, δ, of a material conductor for a particular wave, is the distance into the medium at which the wave amplitude reaches e^{-1} of its value at the surface. Clearly, in terms of the above, this occurs when

or

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \tag{3.69}$$

Notice that $\delta \propto 1/\sqrt{f}$, so as frequency increases (for fixed medium parameters μ and σ) a wave travelling in a good conducting medium will have most of its energy confined to smaller and smaller regions of the conductor cross section. This has important implication for sizing conductors and for isolating electrical components in engineering applications where time-varying fields are involved. Since reducing δ means reducing the region in which most of the field propagates, this parameter is also important in determining useful distances of e-m propagation in applications such as radar.

Example: For seawater, $\sigma = 4 = \mho$/m (nominally) and $\mu = \mu_0$. Determine the effective skin depth for a 10 MHz high frequency radar signal travelling over the ocean surface.

More IMPORTANT stuff about SKIN DEPTH appears on a related Tutorial.

Intrinsic Impedance in Terms of Skin Depth for Good Conductors:

In this case,

$$\begin{aligned}\eta &\approx \\ &= \\ &= \\ \eta &\approx \end{aligned} \tag{3.70}$$

which means: Magnitude: ; Phase:

Thus, while (3.68) becomes

$$\tag{3.71}$$

from equation (3.57),

$$\tag{3.72}$$

<u>Power Considerations in Terms of Skin depth:</u>

For the average power density (Poynting vector) we had from (3.60)

For the good conductor we therefore have

or

$$\vec{\mathcal{P}}_{\text{av}} = < \vec{\mathcal{P}} > = \frac{1}{4}\sigma\delta E_{x0}^{+2} e^{-2z/\delta}\,\hat{z} \ \ \text{W/m}^2 \tag{3.73}$$

Thus, at one skin depth (i.e. when $z = \delta$), the power density is only e^{-2} of its value at the surface (i.e. where $z = 0$).

<u>Dissipated Power:</u> Equation (3.73) represents the average power per unit area being transported through the good conductor. We now ask, "What about the dissipated power because of the lossy character of the conductor?" We concluded in Section 3.3 that the power loss per volume is characterized by the dot product $\underset{\sim}{\vec{E}} \cdot \underset{\sim}{\vec{J}}_c$ in watts per cubic metre. Lets symbolize this by P_{d_L}:

The time average of this quantity is then

$$P_{d_{L\mathrm{avg}}} =< P_{d_L} >= \frac{\sigma}{2}\left(E_{x0}^{+^2}\right) e^{-2z/\delta} \ \ \mathrm{W/m^3} \tag{3.74}$$

Consider a volume V as shown and let's develop the average power, $< P_L >$, dissipated within the volume based on the average power loss density, $< P_{d_L} >$ in (3.74).

$$\text{Average power loss in } V = \int_V (\text{average power loss density})$$

That is,

(3.75)

Let's check this by considering the difference between the average power entering the volume at z_1 and that exiting it at z_2. To do this we must integrate (3.73) for the power flow per square metre within the region for the two values of z.

as in (3.75). Therefore, we now have shown that

Avg. Power loss within volume = Avg. Power entering – Avg. Power exiting

3.6 Reflection and Transmission of Plane Waves

To this point in the analysis, we have considered plane waves travelling in a single medium of infinite extent. We shall now consider what happens when such waves encounter a boundary between two media. In this course, we shall consider only the case of the E-field being in the plane of incidence (this is referred to as *parallel polarization*) where the plane of incidence, in general, is spanned by the propagation wave vector and the normal to the boundary. Furthermore, here we consider only normal incidence – i.e. the propagation vector is perpendicular to the plane of the boundary. While this is somewhat restrictive, it is extremely useful for such applications as e-m waves travelling on two-wire transmission lines. More general angles of incidence will be addressed in a future course. An additional simplification which we make here is the imposition of *linear polarization* in which at any position in the space considered, the tip of the E-field vector traces out a line as time progresses. In fact, we shall take a special case of linear polarization in which the E-field has only an $\hat{x}$ component as before. With these stipulations, we may represent our (special case) E and H fields as follows:

Let $\vec{E}_i$ be the phasor form of the *incident* E-field impinging the media interface as shown:

We choose

$$\vec{E}_i = E_{x0}^+ e^{-\gamma_1 z}\hat{x} \tag{3.76}$$

with $\gamma_1 = \alpha_1 + j\beta_1$ being the propagation constant in the first medium. Immediately, from Maxwell's equations, the corresponding H-field becomes

$$\vec{H}_i = \frac{E_{x0}^+}{\eta_1} e^{-\gamma_1 z}\hat{y} \tag{3.77}$$

where, in general, the intrinsic impedance, η_1, of the first medium may be complex (i.e. if $\sigma \neq 0$).

When the incident wave strikes the boundary at $z = 0$, the wave energy may be partially reflected and partially transmitted. We define a reflection coefficient, Γ, as the ratio of the amplitudes of the reflected and incident electric fields at the boundary:

$$|\vec{E}_R|_{z=0} = \Gamma|\vec{E}_i|_{z=0} \ . \tag{3.78}$$

Similarly, the transmission coefficient, τ, may be defined as the ratio of the amplitudes of the transmitted and incident electric fields at the boundary:

$$|\vec{E}_T|_{z=0} = \tau|\vec{E}_i|_{z=0} \ . \tag{3.79}$$

It is straightforward from Maxwell's equations and the Poynting vector to show that the reflected wave travels in the $-\hat{z}$ direction and the transmitted wave travels in the $+\hat{z}$ direction (for the case of normal incidence). The resulting E- and H-fields, in phasor form, based on (3.76) and(3.77) become

$$\vec{E}_R = \Gamma E_{x0}^+ e^{+\gamma_1 z}\hat{x} = E_{x0}^- e^{+\gamma_1 z}\hat{x} \tag{3.80}$$

$$\vec{E}_T = \tau E_{x0}^+ e^{-\gamma_2 z}\hat{x} \tag{3.81}$$

$$\vec{H}_R = \frac{\Gamma E_{x0}^+}{\eta_1} e^{+\gamma_1 z}(-\hat{y}) = \frac{E_{x0}^-}{\eta_1} e^{+\gamma_1 z}(-\hat{y}) \tag{3.82}$$

$$\vec{H}_T = \frac{\tau E_{x0}^+}{\eta_2} e^{-\gamma_z z}(\hat{y}) \tag{3.83}$$

It should not be surprising that Γ and τ can be expressed in terms of the intrinsic impedances, η_1 and η_2, of the media involved. We shall show this by applying the boundary conditions (at $z = 0$). First, we had the fact that the tangential electric field must be continuous across the boundary. Our choice of $\vec{E}_i$, in fact, has only a tangential component (see diagram above), and we have from (3.76), (3.80) and (3.81) that

This implies that

$$\boxed{1 + \Gamma = \tau} \tag{3.84}$$

Next, applying the magnetic field boundary conditions (with no surface current density), the tangential H-field must also be continuous across the boundary:
From (3.77), (3.82) and (3.83)

or

$$\boxed{\frac{1}{\eta_1} - \frac{\Gamma}{\eta_1} = \frac{\tau}{\eta_2}} \tag{3.85}$$

From (3.84) and (3.85) it may be trivially verified that

$$\boxed{\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}} \tag{3.86}$$

and

$$\boxed{\tau = 1 + \Gamma = \frac{2\eta_2}{\eta_2 + \eta_1}} \tag{3.87}$$

Special Cases:

Consider, next, the reflection/transmission characteristics of a variety of important cases.

1. Medium 2 is a Perfect Conductor:

   (For this case, the first medium could be free space). For the perfect conductor, $\sigma_2 \longrightarrow \infty$. Then, which again implies that there is no time-varying E-field inside a perfect conductor, and from (3.80) the reflected electric field becomes

(3.88)

Therefore, at $z = 0$, on comparing (3.76) and (3.88), we see that there is a 180° phase reversal on reflection from the perfect conductor. However, from (3.77) and (3.82), given that $\Gamma = -1$,

$$\vec{H}_R = \vec{H}_i$$

and there is no phase reversal of the H-field on reflection from the perfect conductor.

Further to this special case, let's assume that medium 1 is a perfect dielectric (i.e. $\sigma = 0$) so that from our earlier deliberations, the propagation constant is imaginary – i.e., $\gamma_1 = \alpha_1 + j\beta_1 = 0 + j\beta_1$. In view of (3.76) and (3.88), we may write the total phasor E-field in medium 1 as

In the time domain,

or

$$\underset{\sim}{\vec{E}}_1 = 2E^+_{x0} \sin \beta_1 z \sin \omega t \, \hat{x} \ . \tag{3.89}$$

The amplitude of this total E-field will be maximum at positions given by

$$|\sin \beta_1 z| = 1$$

and, recalling that $z < 0$ in the first medium,

$$\beta_1 z = -\left(\frac{2n+1}{2}\right)\pi$$

where n belongs to the whole numbers. Therefore, the maxima in the amplitude of $\underset{\sim}{\vec{E}}_1$ occur at

Similarly, minimum values of the $\underset{\sim}{\vec{E}}$ amplitude of (3.89) occur at

$$|\sin \beta_1 z| = 0$$

i.e.

At these positions, for the case at hand, the $|\underset{\sim}{\vec{E}}|$-minima are, in fact, zero.

Sketch:

Clearly, equation (3.89) represents a *standing wave* for the total $\underset{\sim}{\vec{E}}$-field in the medium containing the incident and reflected waves when medium 1 is a perfect dielectric and medium 2 is a perfect conductor.

Standing Wave Ratio (SWR)

In general the standing wave ration (SWR) symbolized by the letter s is defined as

$$s = \frac{|\underset{\sim}{\vec{E}}|_{\max}}{|\underset{\sim}{\vec{E}}|_{\min}} \tag{3.90}$$

where $\underset{\sim}{\vec{E}}$ is the total E-field in which the standing wave exists. Clearly, when medium 2 is a perfect conductor and medium 1 has an imaginary γ (as is the case for a perfect dielectric) we see that

(i.e., $s \rightarrow \infty$ means that the incident wave energy is totally reflected from the boundary and $\tau = 0$). We shall treat the SWR more generally shortly. It is indeed a parameter of great practical importance in any application involving the transfer of time-varying e-m energy from one region to another of differing electric/magnetic properties.

2. "Matched" Media:

   Consider, next, the case where medium 1 and medium 2 have the same intrinsic impedance (i.e., $\eta_1 = \eta_2$). We say that the media are "impedance matched". From equations (3.86) and (3.87) we immediately have

$$\Gamma = 0 \quad \textit{and} \quad \tau = 1 \; .$$

   For these values, we note that all of the incident energy is transmitted into medium 2 and, therefore, no standing wave exists in medium 1. The amplitudes of the E-fields are as shown (if the media are NOT dissipative).

   Clearly, from (3.90), $s \rightarrow 1$. From the two cases considered, we might expect, as is the case, that the SWR lies in the range given by

$$1 \leq s < \infty$$

with $s = 1$ being the case for perfectly matched media and $s \to \infty$ if medium 2 is a perfect conductor when medium 1 has $\sigma = 0$.

Standing Wave Ratio – A more General Approach

We now consider the situation in which medium 1 is a *perfect dielectric*, but medium 2 can be any material. This means that

Then equations (3.76), (3.80), and (3.81) may be written as

$$\vec{E}_i = E_{x0}^+ e^{-j\beta_1 z} \hat{x} \tag{3.91}$$

$$\vec{E}_R = \Gamma E_{x0}^+ e^{j\beta_1 z} \hat{x} \tag{3.92}$$

$$\vec{E}_T = \tau E_{x0}^+ e^{-\alpha_2 z} e^{-j\beta_2 z} \hat{x} \tag{3.93}$$

and equations (3.77), (3.82) and (3.83) are correspondingly

$$\vec{H}_i = \frac{E_{x0}^+}{\eta_1} e^{-j\beta_1 z} \hat{y} \tag{3.94}$$

$$\vec{H}_R = -\frac{\Gamma E_{x0}^+}{\eta_1} e^{j\beta_1 z} \hat{y} \tag{3.95}$$

$$\vec{H}_T = \frac{\tau E_{x0}^+}{\eta_2} e^{-\alpha_2 z} e^{-j\beta_2 z} \hat{y} \tag{3.96}$$

Furthermore,

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad (3.86) \text{ and } \quad \tau = \frac{2\eta_2}{\eta_2 + \eta_1} \quad (3.87)$$

In general, Γ is complex. The total E- and H-fields in region 1 are found, as before, from the sum of the incident and reflected fields to get

(3.97)

and

(3.98)

Equations (3.97) and (3.98) differ from the earlier special cases by virtue of the general reflection coefficient, Γ. We wish to determine the SWR, s, in terms of this Γ.

[We note, in passing, that unlike the case of reflection from a perfect conductor where medium 1 contained only a standing wave, in this more general case both a standing wave and a travelling wave will exist in that medium. This would be easy to see by taking (3.97) to the time domain – DO THIS AS AN EXERCISE!]

Because we desire to shortly determine how the SWR is related to power transmission across a boundary, we proceed as follows:

Squaring the magnitude of $\vec{E}_1$ in equation (3.97) we get

Therefore, using $\Gamma = |\Gamma|e^{j\theta_\Gamma}$ where θ_Γ is the phase angle of Γ,

$$|\vec{E}_1|^2 = |E_{x0}^+|^2 \left\{1 + |\Gamma|^2 + 2|\Gamma| \cos(2\beta_1 z + \theta_\Gamma)\right\}. \tag{3.99}$$

Now, consider the maxima and minima of $|\vec{E}_1|^2$ as given in (3.99):

<u>Maxima:</u> These will clearly occur when

$$\cos(2\beta_1 z + \theta_\Gamma) = {}^{+}1$$

Therefore,

$$|\vec{E}_1|_{\max} = |E_{x0}^+| \, (1 + |\Gamma|) \tag{3.100}$$

<u>Minima:</u> Again, it is obvious that these occur when in (3.99)

$$\cos(2\beta_1 z + \theta_\Gamma) = {}^{-}1$$

in which case

$$|\vec{E}_1|_{\text{min}} = |E^+_{x0}|\,(1 - |\Gamma|) \tag{3.101}$$

Therefore, from equation (3.90),

$$s = \frac{|\underset{\sim}{\vec{E}}|\text{max}}{|\underset{\sim}{\vec{E}}|\text{min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}\ . \tag{3.102}$$

YOU should show that for this case (i.e., medium 1 being a perfect dielectric and medium 2 being general) the $\vec{H}_1$-field is maximum when the $\vec{E}_1$-field is minimum and the $\vec{H}_1$-field is minimum when the $\vec{E}_1$-field is maximum. That is begin by showing

$$|\vec{H}_1|^2 = \frac{|E^+_{x0}|^2}{\eta_1^2}\left\{1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta_1 z + \theta_\Gamma)\right\}. \tag{3.103}$$

To verify that the above analysis reduces to our earlier special cases, consider the situation where medium 2 is a perfect <u>conductor</u> so that $\eta_2 = 0$. Now, from equation (3.86)

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

which implies, for this case,

but because $z < 0$ in medium 1, we'll need

Now, minima in $|\vec{E}_1|$ occur at z values satisfying

$$\cos(2\beta_1 z + \theta_\Gamma) = -1\ \ .$$

This implies

Therefore, $|\vec{E}_1|_{\text{min}}$ occur at $z = -\dfrac{n\lambda_1}{2}$, $n = 0, 1, 2, 3, \ldots$ as before.

As HOMEWORK, show that, for this case, $|\vec{E}_1|_{\text{max}}$ occur at $z = -\dfrac{(2n+1)\lambda_1}{4}$ with $n = 0, 1, 2, 3, \ldots$ as before.

Illustration:

Standing Wave Ratio and Power Transfer Considerations

The SWR, s, is an important practical parameter in power transmission applications. It can be related to the incident and reflected power densities, $\mathcal{P}_i$ and $\mathcal{P}_r$, respectively. We know from Poynting's theorem that

Therefore, the power density ratio, p_r say, may be written as

which implies that

$$|\Gamma| = \sqrt{p_r}$$

and equation (3.102) becomes

$$s = \frac{1 + \sqrt{p_r}}{1 - \sqrt{p_r}} \tag{3.104}$$

or

$$p_r = \left(\frac{s-1}{s+1}\right)^2 \tag{3.105}$$

In general, in practical applications, it is desirable to have as much incident power as possible absorbed by the load – i.e. reflected power should be minimized. To accomplish this, it is desirable to have $s \to 1$ or, equivalently, $\Gamma \to 0$. Again, as noted earlier, the process of accomplishing this is called "impedance matching" and from (3.86) it is clear that perfect matching will require that $\eta_2 = \eta_1$. This may be achieved in a variety of ways, depending on the application. In the next unit, we examine the process in detail for two-wire transmission lines.