

Unit 4

Sinusoidal Steady-State Analysis

In this unit, we consider circuits in which the sources are sinusoidal in nature. The review section of this unit covers most of section 9.1–9.9 of the text. The new material is almost exclusively contained in Chapter 10 of the text.

4.1 Review

4.1.1 Sinusoidal Sources

Up to this point in the course, we have considered only dc sources. Now we consider voltage or current sources that vary sinusoidally with time. For example, if $v(t)$ is a sinusoidal (or co-sinusoidal) voltage it may be written as (dropping the explicit time argument)

$$v = V_m \cos(\omega t + \phi) \quad (4.1)$$

where V_m is the magnitude of the voltage, ω is the radian frequency in radians/s which is related to the frequency f in hertz via

$$\omega = 2\pi f = 2\pi/T \quad (4.2)$$

and ϕ is the so-called *phase angle* measured in radians. Of course, T is the period of the voltage. Notice that a positive ϕ shifts the time function to the left and a negative ϕ shifts the function to the right.

Illustration:

The root mean square (rms) value of a periodic function is simply the “square root of the mean value of the squared function”. Thus, for v as in equation (4.1), the

rms value V_{rms} is given by

$$V_{\text{rms}} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi) dt}$$

from which it is easily shown (DO IT) that

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}} \quad (4.3)$$

Of course, equations completely analogous to the above may be written for sinusoidal current $i(t)$.

4.1.2 Sinusoidal Response

To illustrate the ideas associate with circuit responses to sinusoidal sources consider the following circuit:

Given that the voltage source is sinusoidal as in equation (1), applying KVL to the circuit gives

$$L \frac{di}{dt} + Ri = V_m \cos(\omega t + \phi) \quad (4.4)$$

This is a little more difficult non-homogeneous first order equation than what we had for the dc case. However, from our earlier studies it is not too hard to believe that when the switch is closed there will be a transient response which eventually “settles down” (at least for stable systems) to some steady state value. That is, it will be of the form

$$i = Ae^{-t/\tau} + B \cos(\omega t + \phi - \theta) \quad (4.5)$$

where for linear circuit elements, we have allowed for a magnitude change (as is incorporated in the constant B) and a phase shift as is indicated by the parameter θ . Note that **the frequency of the steady-state response is the same as that of the source** but, in general, the **amplitude and phase angle of the response are different from those of the source**. The first term is transient and the constants A , τ , B , and θ depend on the type and value of the circuit elements. In the RL circuit shown, $\tau = L/R$ as before (i.e. the time constant). By substituting the steady state solution into the differential equation, the coefficient B is easily shown to be

$$B = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}$$

while for the given circuit

$$\theta = \tan^{-1} \left(\frac{\omega L}{R} \right) .$$

Since for this circuit, $i = 0$ at $t = 0$, it is easily deduced that

$$A = -B \cos(\phi - \theta) .$$

4.1.3 Phasors

It is clearly the case that if

$$v = V_m \cos(\omega t + \phi)$$

with V_m being real, then

$$\begin{aligned} v &= \mathcal{R} \{ V_m e^{j(\omega t + \phi)} \} \\ &= \mathcal{R} \{ V_m e^{j\phi} e^{j\omega t} \} \\ &= \mathcal{R} \{ V e^{j\omega t} \} \end{aligned}$$

where

$$V = V_m e^{j\phi} \tag{4.6}$$

is a complex number which is referred to as the **phasor representation** of the sinusoidal (or time-harmonic) time function. Thus, a **phasor** whose magnitude is the amplitude of the time function and whose phase is the phase angle of the time function is a means of representing the function (in the complex domain) without using time. Manipulating functions in this way is also referred to as **frequency-domain analysis**. It is easy to see that a time derivative is transformed to a $j\omega$ in the frequency domain:

Illustration of a Phasor Transform and an Inverse Phasor Transform:

4.1.4 Passive Circuit Elements in Frequency Domain

Resistance

The sign convention for a resistance carrying a sinusoidal current is illustrated as

Considering now that v and i are both sinusoids, we write

$$v = iR$$

which in the frequency domain transforms to

$$V = IR \tag{4.7}$$

where V and I are the phasor representations of v and i , respectively.

Inductance

The sign convention for an inductor carrying a sinusoidal current is illustrated as

Recall the voltage-current relationship for an inductor (where now both are sinusoids):

$$v = L \frac{di}{dt}$$

Since the time derivative transforms to $j\omega$, the equation in phasor form becomes

$$V = L(j\omega I) = j\omega LI \tag{4.8}$$

The quantity $j\omega L$ is referred to as the **impedance** of the inductor and ωL alone is referred to as the **reactance**.

Consider a current given by $i = I_m \cos(\omega t + \theta_i)$ in the inductor. The phasor is

Thus, the time domain voltage is given by

$$v = \qquad \qquad \qquad = \tag{4.9}$$

and we see that the voltage across the inductor *leads* the current through it by 90° .

Capacitance

The sign convention for a capacitor across which exists a sinusoidal voltage is illustrated as

Recall the voltage-current relationship for a capacitor (where now both are sinusoids):

$$i = C \frac{dv}{dt}$$

Since the time derivative transforms to $j\omega$, the equation in phasor form becomes

$$I = C(j\omega V) \rightarrow V = \frac{I}{j\omega C} \quad (4.10)$$

The quantity $1/(j\omega C)$ is referred to as the **impedance** of the capacitor and $-1/(\omega C)$ alone is referred to as the **reactance**. It is easy to show that the voltage across the capacitor *lags* the current through it by 90° (DO THIS). The voltage and current relationships for the three passive elements above may be illustrated by a “phasor” diagram where we have used subscripts to indicate the voltages across the particular elements.

Impedance – General

From equations (4.7), (4.8) and (4.10) we may write generally that

$$V = IZ \quad (4.11)$$

where Z in ohms is the **impedance** of the circuit or circuit element. In general, Z is complex – its real part is “resistance” and its imaginary part is “reactance” –

$$Z = R + jX \quad (4.12)$$

We also define the reciprocal of the impedance as the **admittance** Y in siemens or mhos:

$$Y = \frac{1}{Z} = G + jB \quad (4.13)$$

G is conductance and B is susceptance.

4.2 Kirchhoff's Laws and Impedance Combinations

We will now consider Kirchhoff's voltage and current laws in the phasor (frequency) domain.

Kirchhoff's Voltage Law

It is easy to show in the same manner as considered below for the current law that if sinusoidal voltages v_1, v_2, \dots, v_n exist around a closed path (assuming steady state) in a circuit so that

$$v_1 + v_2 + \dots + v_n = 0 ,$$

then, the phasor equivalent is given by

$$V_1 + V_2 + \dots + V_n = 0 \tag{4.14}$$

where $v_1 \leftrightarrow V_1$, $v_2 \leftrightarrow V_2$, etc..

Kirchhoff's Current Law

Applying KCL to n sinusoidal currents having, in general, different magnitudes and phases (but the same frequency) we get

$$i_1 + i_2 + \dots + i_n = 0 ,$$

In general, each current has the form $i = I_m \cos(\omega t + \theta)$. Now,

$$i_1 = \mathcal{R} \{ I_{m_1} e^{j\theta_1} e^{j\omega t} \} \quad ; \quad i_2 = \mathcal{R} \{ I_{m_2} e^{j\theta_2} e^{j\omega t} \} \quad \text{etc.}$$

Therefore,

which implies

$$I_1 + I_2 + \dots + I_n = 0 . \tag{4.15}$$

Series Impedances

Consider a series combination of complex impedances Z_1, Z_2, \dots, Z_n represented as shown with voltage and current in phasor form.

From equation (4.11) [i.e. $V = IZ$] and KVL,

Therefore,

$$Z_{ab} = Z_s = Z_1 + Z_2 + \dots + Z_n \quad (4.16)$$

where Z_s refers to the *total* series impedance.

Parallel Impedances

Consider a parallel combination of complex impedances Z_1, Z_2, \dots, Z_n represented as shown with voltage and currents in phasor form.

From KCL,

$$I_1 + I_2 + \dots + I_n = I$$

From equation (4.11),

where Z_p is the *total* or equivalent parallel impedance. Therefore,

$$\frac{1}{Z_p} = \frac{1}{Z_1} + \frac{1}{Z_2} + \dots + \frac{1}{Z_n} . \quad (4.17)$$

For example, for the special case of two parallel impedances,

$$Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2} .$$

Delta-Wye Transformations

Just as we have seen that series and parallel impedances add in the same fashion as their resistive counterparts, so the delta-wye transformations on page 9 of Unit 1 of these notes can be used to write an analogous set of transformations involving impedances. The results are given below (the derivations are straightforward as we saw for resistances and won't be repeated here). Consider the following delta-wye configuration:

These will be useful in analyzing bridge-type circuits and also in later courses where 3-phase power is considered.

Summary of the Phasor Approach to Circuit Analysis

For circuits containing passive elements R , L , and C and sinusoidal sources, the phasor approach to circuit analysis is:

1. Convert voltages v and currents i to phasors V and I , respectively.
2. Convert R 's, L 's and C 's to impedances.
3. Use the rules of circuit analysis to manipulate the circuit in the phasor domain.
4. Return to the time domain for voltages and currents etc. by using the inverse-phasor-transformation forms $i = \mathcal{R} \{ I e^{j\omega t} \}$ and $v = \mathcal{R} \{ V e^{j\omega t} \}$.

Page 9 should be a looseleaf sheet with Drill problem 9.13 from the text.

4.2.1 Phasor Diagrams

As noted in Section 4.1.3, a phasor may be represented geometrically in the complex plane as an arrow whose magnitude is the magnitude of the phasor and whose direction is the phase angle. Such phasors may represent voltages, currents or impedances. To illustrate their usefulness, consider the following example:

Example of Using a Phasor Diagram: Consider the circuit shown below. Assume the phase angle of the voltage phasor V is 0° (i.e. $V = V_m \angle 0^\circ$). Use a phasor diagram to determine the value of R that will cause the current through the resistor to *lag* the source current by 45° when the radian frequency of the sinusoids involved is 2500 rad/s .

4.2.2 Useful Circuit Analysis Techniques – Phasor Domain

In our earlier analyses for circuits with dc sources, we noted that the following techniques:

- (1) Source Transformation
- (2) Thévenin/Norton Equivalent Circuits
- (3) Node-voltage Analysis
- (4) Mesh-current Analysis

These techniques may be also applied in steady-state ac analysis in the phasor domain. Since the corresponding phasor analysis requires no fundamentally new knowledge in order to apply these methods, we will consider them by way of examples.

Example 1:

4.3 Power Calculations (Sinusoidal, Steady State)

Read Chapter 10 of the text.

The sign convention associated with the equation

$$p = vi \tag{4.18}$$

for **instantaneous power** is as shown. As before, a value $p > 0$ means power is being dissipated (like in a resistor) and $p < 0$ means power is being supplied to the circuit. Here v and i are sinusoidal steady-state signals which may be written as

$$v = V_m \cos(\omega t + \theta_v) \tag{4.19}$$

$$i = I_m \cos(\omega t + \theta_i) \tag{4.20}$$

It is conventional to choose zero time when the current is passing through a positive maximum so if we set

$$v = V_m \cos(\omega t + \theta_v - \theta_i) \tag{4.21}$$

$$i = I_m \cos(\omega t) \tag{4.22}$$

we see that the common shift of phase equal to $-\theta_i$ (i.e. setting the phase shift of the current to zero) results in the same relative phase between the current and the voltage. Let's define

$$\Delta\theta = \theta_v - \theta_i$$

so that equations (4.21) and (4.22) may be written as

$$v = V_m \cos(\omega t + \Delta\theta) \tag{4.23}$$

$$i = I_m \cos(\omega t) . \tag{4.24}$$

We have already seen that

for a pure inductance, $\Delta\theta = 90^\circ$ (voltage leads current or current lags voltage);

for a pure capacitance, $\Delta\theta = -90^\circ$ (voltage lags current or current leads voltage);,

for a pure resistance, $\Delta\theta = 0^\circ$ (voltage and current are in phase).

On the basis of (4.18), (4.23) and (4.24) we have for *instantaneous* power

$$p = vi = V_m \cos(\omega t + \Delta\theta) I_m \cos(\omega t)$$

Notice that the first term is independent of time while the remaining two terms are time-harmonic. It is easy to show (DO THIS), that when the last two terms are averaged over one period (T), the result is zero. The first term is clearly the *average power*. That is average power P is given by

$$P = \frac{V_m I_m}{2} \cos \Delta\theta \quad (4.26)$$

This average power is the *real power* in the circuit and its unit is watts (W).

Suppose next that we define

$$Q = \frac{V_m I_m}{2} \sin \Delta\theta \quad (4.27)$$

Then, equation (4.25) may be written as

$$p = P + P \cos 2\omega t - Q \sin 2\omega t \quad (4.28)$$

The quantity Q in (4.27) is referred to as **reactive power**. It is power being “stored” or “released” in capacitors or inductors and as we have indicated above it averages to zero over a period. The unit on this reactive power is the VAR (volt-amp reactive).

Let’s consider the implications of equation (4.25) for R , L and C elements while remembering the definitions of P and Q :

Pure Resistance

Here, $\Delta\theta = 0^\circ \Rightarrow Q = 0$ and from (4.28) $p =$:

Pure Inductance

Here, $\Delta\theta = 90^\circ \Rightarrow P = 0$ and from (4.28) $p(t) =$.

(See Illustration below). As we intimated, the average power is zero in this case and the inductor alternately stores power ($p > 0$) or releases power ($p < 0$). Clearly, also $Q > 0$ (see equation (4.27) and note here $\Delta\theta = 90^\circ$).

Pure Capacitance

Here, $\Delta\theta = -90^\circ \Rightarrow P = 0$ and from (4.28) $p(t) =$.

Again, the average power is zero in this case and the capacitor alternately stores power ($p > 0$) or releases power ($p < 0$). Clearly, also $Q < 0$ (see equation (4.27) and note here $\Delta\theta = -90^\circ$). (See Illustration below).

The summary terminology says that if $Q > 0$ (i.e. an inductor in view) the circuit absorbs magnetizing vars and if $Q < 0$ (i.e. a capacitor in view) the circuit delivers magnetizing vars.

Power Factor

The quantity $\Delta\theta$ is referred to as the **power factor angle**.

The quantity $\cos \Delta\theta$ is referred to as the **power factor**.

The quantity $\sin \Delta\theta$ is referred to as the **reactive factor**.

Of course, since in general, $\cos \alpha = \cos -\alpha$, knowing the power factor only allows knowledge of an ambiguous power factor angle. Therefore, we invent the following terminology:

lagging power factor meaning current lags voltage (i.e. an inductive load);

leading power factor meaning current leads voltage (i.e. a capacitive load).

Example: Consider that the voltage and current associated with the terminals of the network shown are given by

$$v = 75 \cos \omega(\omega t - 15^\circ) \text{ V}$$

$$i = 16 \sin(\omega t + 60^\circ) \text{ A}$$

Determine (i) the reactive power and the average power at the terminals of the network; (ii) whether the network inside the box is absorbing or delivering average power and (iii) whether the network inside the box is absorbing or delivering magnetizing vars. (iv) Determine the power factor and the reactive factor for the network inside the box.

4.3.1 The Relationship Between RMS Values and Average Power:

We have seen from equation (4.26) that the average power is given by

$$P = \frac{V_m I_m}{2} \cos \Delta\theta .$$

Clearly, this may be written as

$$P = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos \Delta\theta$$

or

$$P = \tag{4.29}$$

V_{rms} and I_{rms} are sometimes referred to as *effective* values V_{eff} and I_{eff} , respectively. The reason for this terminology is as follows: If a dc voltage V is applied to a resistance

R for a time T , the energy dissipated is the same as would be for a sinusoidal voltage whose rms value is equivalent to V when that source is connected to an equivalent R for time T . [YOU SHOULD PROVE THIS].

Arguing in the same fashion as above, the reactive power is given by

$$Q = V_{\text{rms}}I_{\text{rms}} \sin \Delta\theta = V_{\text{eff}}I_{\text{eff}} \sin \Delta\theta \quad (4.30)$$

This relationship between average power and rms values can be argued from the definition of average power. For example the average power dissipated in a resistance R over when the latter carries a current $i = I_m \cos(\omega t + \theta_i)$ is given simply by

$$P =$$

or

$$P = I_{\text{rms}}^2 R . \quad (4.31)$$

Similarly, it is easily deduced that

$$P = \frac{V_{\text{rms}}^2}{R} . \quad (4.32)$$

An example (Drill Exercise 10.5, page 500 of text):

Here we use the rms value of a non-sinusoidal current to determine power delivered to a 5Ω resistance by that current. The current waveform is as shown and its peak value is 180 mA.

4.3.2 Complex Power

We define a value referred to as complex power by the relationship

$$S = P + jQ \quad (4.33)$$

where P and Q are still the real and reactive power. The unit for S is the **volt-amp** and this unit terminology is used to distinguish it from watts (for P) and vars (for Q). S may be represented in the form of a “power triangle”.

$$\tan \Delta\theta = \frac{Q}{P} \quad (4.34)$$

and the magnitude of S is clearly given by

$$|S| = \sqrt{P^2 + Q^2} \quad (4.35)$$

This magnitude of the complex power is referred to as the **apparent power** because it represents the required volt-amp capacity of a device required to supply a certain average power. If a load is not purely resistive, this apparent power is always greater than the average power – that is, the amount of power which must be supplied to a system is always greater than what the system is able to output ($|S| > P$).

Complex Power, RMS Values and RMS Phasors

The complex power is nicely related to rms values of voltage and current as shown below:

Thus,

$$S = V_{\text{rms}} I_{\text{rms}} e^{j\Delta\theta} \quad (4.36)$$