Chapter 4

Sinusoidal Steady-State Analysis

In this unit, we consider circuits in which the sources are sinusoidal in nature. The review section of this unit covers most of section 9.1–9.9 of the text. The new material is almost exclusively contained in Chapter 10 of the text.

4.1 Review

4.1.1 Sinusoidal Sources

Up to this point in the course, we have considered only dc sources. Now we consider voltage or current sources that vary sinsusoidally with time. For example, if $v(t)$ is a sinusoidal (or co-sinusoidal) voltage it may be written as (dropping the explicit time argument)

$$v = V_m \cos(\omega t + \phi) \tag{4.1}$$

where V_m is the magnitude of the voltage, ω is the radian frequency in radians/s which is related to the frequency f in hertz via

$$\omega = 2\pi f = 2\pi / T \tag{4.2}$$

and ϕ is the so-called *phase angle* measured in radians. Of course, T is the period of the voltage. Notice that a positive ϕ shifts the time function to the left and a negative ϕ shifts the function to the right.
Illustration:

The root mean square (rms) value of a periodic function is simply the "square root of the mean value of the squared function". Thus, for v as in equation (4.1), the rms value V_{rms} is given by

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi) dt}$$

from which it is easily shown (DO IT) that

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}} \tag{4.3}$$

Of course, equations completely analogous to the above may be written for sinusoidal current $i(t)$.

4.1.2 Sinusoidal Response

To illustrate the ideas associate with circuit responses to sinusoidal sources consider the following circuit:

Given that the voltage source is sinusoidal as in equation (1), applying KVL to the circuit gives

$$L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi) \tag{4.4}$$

This is a little more difficult non-homogeneous first order equation than what we had for the dc case. However, from our earlier studies it is not too hard to believe that when the switch is closed there will be a transient response which eventually "settles down" (at least for stable systems) to some steady state value. That is, it will be of the form

$$i = Ae^{-t/\tau} + B\cos(\omega t + \phi - \theta) \tag{4.5}$$

where for linear circuit elements, we have allowed for a magnitude change (as is incorporated in the constant B) and a phase shift as is indicated by the parameter θ. Note that **the frequency of the steady-state response is the same as that of the source** but, in general, the **amplitude and phase angle of the response are different from those of the source**. The first term is transient and the constants A, τ, B, and θ depend on the type and value of the circuit elements. In the RL circuit shown, $\tau = L/R$ as before (i.e. the time constant). By substituting the steady state solution into the differential equation, the coefficient B is easily shown to be

$$B = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}$$

while for the given circuit

$$\theta = \tan^{-1}\left(\frac{\omega L}{R}\right) \quad .$$

Since for this circuit, $i = 0$ at $t = 0$, it is easily deduced that

$$A = -B\cos(\phi - \theta) \quad .$$

4.1.3 Phasors

It is clearly the case that if

$$v = V_m \cos(\omega t + \phi)$$

with V_m being real, then

$$\begin{aligned} v &= \mathcal{R}\left\{V_m e^{j(\omega t + \phi)}\right\} \\ &= \mathcal{R}\left\{V_m e^{j\phi} e^{j(\omega t)}\right\} \\ &= \mathcal{R}\left\{V e^{j(\omega t)}\right\} \end{aligned}$$

where

$$V = V_m e^{j\phi} \tag{4.6}$$

is a complex number which is referred to as the **phasor representation** of the sinusoidal (or time-harmonic) time function. Thus, a **phasor** whose magnitude is the amplitude of the time function and whose phase is the phase angle of the time function is a means of representing the function (in the complex domain) without using time. Manipulating functions in this way is also referred to as **frequency-domain analysis**. It is easy to see that a time derivative is transformed to a $j\omega$ in the frequency domain:

<u>Illustration of a Phasor Transform and an Inverse Phasor Transform:</u>

4.1.4 Passive Circuit Elements in Frequency Domain

Resistance
The sign convention for a resistance carrying a sinusoidal current is illustrated as

Considering now that v and i are both sinusoids, we write

$$v = iR$$

which in the frequency domain transforms to

$$V = IR \tag{4.7}$$

where V and I are the phasor representations of v and i, respectively.

Inductance
The sign convention for an inductor carrying a sinusoidal current is illustrated as

Recall the voltage-current relationship for an inductor (where now both are sinusoids):

$$v = L\frac{di}{dt}$$

Since the time derivative transforms to $j\omega$, the equation in phasor form becomes

$$V = L(j\omega I) = j\omega LI \tag{4.8}$$

The quantity $j\omega L$ is referred to as the **impedance** of the inductor and ωL alone is referred to as the **reactance**.

Consider a current given by $i = I_m \cos(\omega t + \theta_i)$ in the inductor. The phasor is

Thus, the time domain voltage is given by

$$v = \qquad\qquad\qquad = \tag{4.9}$$

and we see that the voltage across the inductor *leads* the current through it by 90°.

Capacitance

The sign convention for a capacitor across which exists a sinusoidal voltage is illustrated as

Recall the voltage-current relationship for a capacitor (where now both are sinsusoids):

$$i = C\frac{dv}{dt}$$

Since the time derivative transforms to $j\omega$, the equation in phasor form becomes

$$I = C(j\omega V) \;\rightarrow\; V = \frac{I}{j\omega C} \tag{4.10}$$

The quantity $1/(j\omega C)$ is referred to as the **impedance** of the capacitor and $-1/(\omega C)$ alone is referred to as the **reactance**. It is easy to show that the voltage across the capacitor *lags* the current through it by 90° (DO THIS). The voltage and current relationships for the three passive elements above may be illustrated by a "phasor" diagram where we have used subscripts to indicate the voltages across the particular elements.

Impedance – General

From equations (4.7), (4.8) and (4.10) we may write generally that

$$V = IZ \tag{4.11}$$

where Z in ohms is the **impedance** of the circuit or circuit element. In general, Z is complex – its real part is "resistance" and its imaginary part is "reactance" –

$$Z = R + jX \tag{4.12}$$

We also define the reciprocal of the impedance as the **admittance** Y in siemens or mhos:

$$Y = \frac{1}{Z} = G + jB \tag{4.13}$$

G is conductance and B is susceptance.

4.2 Kirchhoff's Laws and Impedance Combinations

We will now consider Kirchhoff's voltage and current laws in the phasor (frequency) domain.

Kirchhoff's Voltage Law

It is easy to show in the same manner as considered below for the current law that if sinusoidal voltages v_1,v_2,...,v_n exist around a closed path (assuming steady state) in a circuit so that

$$v_1 + v_2 + \ldots + v_n = 0 \ ,$$

then, the phasor equivalent is given by

$$V_1 + V_2 + \ldots + V_n = 0 \tag{4.14}$$

where $v_1 \leftrightarrow V_1$, $v_2 \leftrightarrow V_2$, etc..

Kirchhoff's Current Law

Applying KCL to n sinusoidal currents having, in general, different magnitudes and phases (but the same frequency) we get

$$i_1 + i_2 + \ldots + i_n = 0 \ ,$$

In general, each current has the form $i = I_m \cos(\omega t + \theta)$. Now,

$$i_1 = \mathcal{R}\left\{I_{m_1} e^{j\theta_1} e^{j\omega t}\right\} \ \ ; \ \ i_2 = \mathcal{R}\left\{I_{m_2} e^{j\theta_2} e^{j\omega t}\right\} \ \ \text{etc.}$$

Therefore,

which implies

$$I_1 + I_2 + \ldots + I_n = 0 \ . \tag{4.15}$$

Series Impedances

Consider a series combination of complex impedances Z_1,Z_2,...,Z_n represented as shown with voltage and current in phasor form.

From equation (4.11) [i.e. $V = IZ$] and KVL,

Therefore,

$$Z_{ab} = Z_s = Z_1 + Z_2 + \ldots + Z_n \tag{4.16}$$

where Z_s refers to the *total* series impedance.

Parallel Impedances

Consider a parallel combination of complex impedances Z_1,Z_2,...,Z_n represented as shown with voltage and currents in phasor form.

From KCL,

$$I_1 + I_2 + \ldots + I_n = I$$

From equation (4.11),

where Z_p is the *total* or equivalent parallel impedance. Therefore,

$$\frac{1}{Z_p} = \frac{1}{Z_1} + \frac{1}{Z_2} + \ldots + \frac{1}{Z_n} \quad . \tag{4.17}$$

For example, for the special case of two parallel impedances,

$$Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2} \quad .$$

Delta-Wye Transformations

Just as we have seen that series and parallel impedances add in the same fashion as their resistive counterparts, so the delta-wye transformations on page 9 of Unit 1 of these notes can be used to write an analogous set of transformations involving impedances. The results are given below (the derivations are straightforward as we saw for resistances and won't be repeated here). Consider the following delta-wye configuration:

These will be useful in analyzing bridge-type circuits and also in later courses where 3-phase power is considered.

Summary of the Phasor Approach to Circuit Analysis

For circuits containing passive elements R, L, and C and sinusoidal sources, the phasor approach to circuit analysis is:

1. Convert voltages v and curernts i to phasors V and I, respectively.

2. Convert R's, L's and C's to impedances.

3. Use the rules of circuit analysis to manipulate the circuit in the phasor domain.

4. Return to the time domain for voltages and currrents etc. by using the inverse-phasor-transformation forms $i = \mathcal{R}\left\{Ie^{j\omega t}\right\}$ and $v = \mathcal{R}\left\{Ve^{j\omega t}\right\}$.

Page 9 should be a looseleaf sheet with a sample problem.

4.2.1 Phasor Diagrams

As noted in Section 4.1.3, a phasor may be represented geometrically in the complex plane as an arrow whose magnitude is the magnitude of the phasor and whose direction is the phase angle. Such phasors may represent voltages, currents or impedances. To illustrate their usefulness, consider the following example:

Example of Using a Phasor Diagram: Consider the circuit shown below. Assume the phase angle of the voltage phasor V is $0°$ (i.e. $V = V_m\angle 0°$). Use a phasor diagram to determine the value of R that will cause the current through the resistor to *lag* the source current by $45°$ when the radian frequency of the sinusoids involved is 2500 rad/s.

4.2.2 Useful Circuit Analysis Techniques – Phasor Domain

In our earlier analyses for circuits with dc sources, we noted that the following techniques:

(1) Source Transformation

(2) Thévenin/Norton Equivalent Circuits

(3) Node-voltage Analysis

(4) Mesh-current Analysis

These techniques may be also applied in steady-state ac analysis in the phasor domain. Since the corresponding phasor analysis requires no fundamentally new knowledge in order to apply these methods, we will consider them by way of examples.

Example 1:

4.3 Power Calculations (Sinusoidal, Steady State)

Read Chapter 10 of the text.

The sign convention associated with the equation

$$p = vi \tag{4.18}$$

for **instantaneous power** is as shown. As before, a value $p > 0$ means power is being dissipated (like in a resistor) and $p < 0$ means power is being supplied to the circuit. Here v and i are sinusoidal steady-state signals which may be written as

$$v = V_m \cos(\omega t + \theta_v) \tag{4.19}$$

$$i = I_m \cos(\omega t + \theta_i) \tag{4.20}$$

It is conventional to choose zero time when the current is passing through a positive maximum so if we set

$$v = V_m \cos(\omega t + \theta_v - \theta_i) \tag{4.21}$$

$$i = I_m \cos(\omega t) \tag{4.22}$$

we see that the common shift of phase equal to $-\theta_i$ (i.e. setting the phase shift of the current to zero) results in the same relative phase between the current and the voltage. Let's define

$$\Delta\theta = \theta_v - \theta_i$$

so that equations (4.21) and (4.22) may be written as

$$v = V_m \cos(\omega t + \Delta\theta) \tag{4.23}$$

$$i = I_m \cos(\omega t) \ . \tag{4.24}$$

We have already seen that
for a pure inductance, $\Delta\theta = 90°$ (voltage leads current or current lags voltage);
for a pure capacitance, $\Delta\theta = -90°$ (voltage lags current or current leads voltage);,
for a pure resistance, $\Delta\theta = 0°$ (voltage and current are in phase).

On the basis of (4.18), (4.23) and (4.24)we have for *instantaneous* power

$$p = vi = V_m \cos(\omega t + \Delta\theta) I_m \cos(\omega t)$$

Notice that the first term is independent of time while the remaining two terms are time-harmonic. It is easy to show (DO THIS), that when the last two terms are averaged over one period (T), the result is zero. The first term is clearly the *average power*. That is average power P is given by

$$P = \frac{V_m I_m}{2} \cos \Delta\theta \tag{4.26}$$

This average power is the *real power* in the circuit and its unit is watts (W).

Suppose next that we define

$$Q = \frac{V_m I_m}{2} \sin \Delta\theta \quad . \tag{4.27}$$

Then, equation (4.25) may be written as

$$p = P + P \cos 2\omega t - Q \sin 2\omega t \tag{4.28}$$

The quantity Q in (4.27) is referred to as **reactive power**. It is power being “stored” or “released” in capacitors or inductors and as we have indicated above it averages to zero over a period. The unit on this reactive power is the VAR (volt-amp reactive).

Let’s consider the implications of equation (4.25) for R, L and C elements while remembering the definitions of P and Q:

Pure Resistance

Here, $\Delta\theta = 0^\circ \Rightarrow Q = 0$ and from (4.28) $p =$:

Pure Inductance

Here, $\Delta\theta = 90^\circ \Rightarrow P = 0$ and from (4.28) $p(t) =$.

(See Illustration below). As we intimated, the average power is zero in this case and the inductor alternately stores power ($p > 0$) or releases power ($p < 0$). Clearly, also $Q > 0$ (see equation (4.27) and note here $\Delta\theta = 90^\circ$).

Pure Capacitance

Here, $\Delta\theta = -90^\circ \Rightarrow P = 0$ and from (4.28)$p(t) =$.

Again, the average power is zero in this case and the capacitor alternately stores power ($p > 0$) or releases power ($p < 0$). Clearly, also $Q < 0$ (see equation (4.27) and note here $\Delta\theta = -90^\circ$).(See Illustration below).

The summary terminology says that if $Q > 0$ (i.e. an inductor in view) the circuit absorbs magnetizing vars and if $Q < 0$ (i.e. a capacitor in view) the circuit delivers magnetizing vars.

Power Factor

The quantity $\Delta\theta$ is referred to as the **power factor angle**.

The quantity $\cos\Delta\theta$ is referred to as the **power factor**.

The quantity $\sin\Delta\theta$ is referred to as the **reactive factor**.

Of course, since in general, $\cos\alpha = \cos -\alpha$, knowing the power factor only allows knowledge of an ambiguous power factor angle. Therefore, we invent the following terminology:

lagging power factor meaning current lags voltage (i.e. an inductive load);

leading power factor meaning current leads voltage (i.e. a capacitive load).

Example: Consider that the voltage and current associated with the terminals of the network shown are given by

$$\begin{aligned} v &= 75\cos\omega(\omega t - 15^\circ)\ \text{V} \\ i &= 16\cos(\omega t + 60^\circ)\ \text{A} \end{aligned}$$

Determine (i) the reactive power and the average power at the terminals of the network; (ii) whether the network inside the box is absorbing or delivering average power and (iii) whether the netwok inside the box is absorbing or delivering magnetizing vars. (iv) Determine the power factor and the reactive factor for the network inside the box.

4.3.1 The Relationship Between RMS Values and Average Power:

We have seen from equation (4.26) that the average power is given by

$$P = \frac{V_m I_m}{2}\cos\Delta\theta\ .$$

Clearly, this may be written as

$$P = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos\Delta\theta$$

or

$$P = \qquad (4.29)$$

V_{rms} and I_{rms} are sometimes referred to as *effective* values V_{eff} and I_{eff}, respectively. The reason for this terminology is as follows: If a dc voltage V is applied to a resistance

R for a time T, the energy dissipated is the same as would be for a sinusoidal voltage whose rms value is equivalent to V when that source is connected to an equivalent R for time T. [YOU SHOULD PROVE THIS].

Arguing in the same fashion as above, the reactive power is given by

$$Q = V_{\text{rms}} I_{\text{rms}} \sin \Delta\theta = V_{\text{eff}} I_{\text{eff}} \sin \Delta\theta \tag{4.30}$$

This relationship between average power and rms values can be argued from the definition of average power. For example the average power dissipated in a resistance R over when the latter carries a current $i = I_m \cos(\omega t + \theta_i)$ is given simply by

$$P =$$

or

$$P = I_{\text{rms}}^2 R \,. \tag{4.31}$$

Similarly, it is easily deduced that

$$P = \frac{V_{\text{rms}}^2}{R} \,. \tag{4.32}$$

<u>An example (Drill Exercise 10.3, page 460 of text):</u>

Here we use the rms value of a non-sinusoidal current to determine power delivered to a 5 Ω resistance by that current. The current waveform is as shown and its peak value is 180 mA.

4.3.2 Complex Power

We define a value referred to as complex power by the relationship

$$S = P + jQ \tag{4.33}$$

where P and Q are still the real and reactive power. The unit for S is the **volt-amp** and this unit terminology is used to distinguish it from watts (for P) and vars (for Q). S may be represented in the form of a "power triangle".

$$\tan \Delta\theta = \frac{Q}{P} \tag{4.34}$$

and the magnitude of S is clearly given by

$$|S| = \sqrt{P^2 + Q^2} \tag{4.35}$$

This magnitude of the complex power is referred to as the **apparent power** because it represents the required volt-amp capacity of a device required to supply a certain average power. If a load is not purely resistive, this apparent power is always greater than the average power – that is, the amount of power which must supplied to a system is always greater than what the system is able to output ($|S| > P$).

Complex Power, RMS Values and RMS Phasors

The complex power is nicely related to rms values of voltage and current as shown below:

Thus,

$$S = V_{\text{rms}} I_{\text{rms}} e^{j\Delta\theta} \tag{4.36}$$

The previous (approximately) two pages (18 and 19) should be on hand-written notes.

4.4 Maximum Power Transfer

Recall that for a resistive circuit as shown

maximum power is transferred to R_L when $R_L = R_{\text{Th}}$ and this power is given by $P_{\text{max}} = V_{\text{Th}}^2/(4R_L)$.

Consider now a similar circuit involving impedances as shown:

We again wish to consider the condition on the load impedance Z_L which is necessary for maximum average power transfer to that load. From the circuit,

$$\bar{I}_{\text{rms}} = \underline{\hspace{6cm}}$$

Then

$$P = \left|\bar{I}_{\text{rms}}\right|^2 R_L = \underline{\hspace{7cm}} \tag{4.43}$$

Here, R_L and X_L are independent variables and the "Th" subscripted variables are constants. To find the values of R_L and X_L which maximize P we set first derivatives to zero and solve as usual; i.e.

Clearly,

$$\frac{\partial P}{\partial X_L} = \underline{\hspace{8cm}} \tag{4.44}$$

and

$$\frac{\partial P}{\partial R_L} = \underline{\hspace{8cm}} \tag{4.45}$$

In (4.44), when $X_L = -X_{\text{Th}}$, $\dfrac{\partial P}{\partial X_L} = 0$.
In (4.45) when $R_L = \sqrt{R_{\text{Th}}^2 + (X_L + X_{\text{Th}})^2}$ (along with the maximizing condition on X_L) then $\partial P/\partial R_L = 0$. That is, both derivatives are zero when

$$R_L = R_{\text{Th}} \quad \text{and} \quad X_L = -X_{\text{Th}} \ ,$$

or

In other words, the condition for maximum power transfer to the load is that the load impedance must be equal to the conjugate of the Thévenin impedance.

$$Z_L = Z_{\text{Th}}^* \tag{4.46}$$

Thus, from equation (4.43)

$$P_{\text{max}} = \underline{\hspace{3cm}} = \underline{\hspace{3cm}} \tag{4.47}$$

Noting that the rms Thévenin voltage is related to the maximum (peak) voltage voltage via

$$P_{\text{max}} = \underline{\hspace{3cm}} \tag{4.48}$$

That is, if peak voltage is used, the condition for maximum power transfer to the load results in (4.48). Note that (4.47) reduces to our earlier result that $R_L = R_{\text{Th}}$ for maximum power transfer when a purely resistive circuit is in view.

4.5 Steady State response as a function of Frequency

4.5.1 Series RLC Circuit

Consider the following series RLC circuit in which the phasor equivalents of v and i are V and I, respectively.

Impedance due to capacitor: $Z_C = -\dfrac{j}{\omega C}$; clearly, as $\omega \to 0$, $Z_c \to \infty$ and $i \to 0$.
Impedance due to inductor: $Z_L = j\omega L$; clearly, as $\omega \to \infty$, $Z_L \to \infty$ and $i \to 0$.

Therefore, circuits containing inductors and capacitors have responses that are *frequency dependent*. First, let's look at

The Current Response:

Using phasors

$$I = \frac{V}{Z} \tag{4.49}$$

For the given circuit, $Z = R - \dfrac{j}{\omega C} + j\omega L$ or

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right) \tag{4.50}$$

This value of Z can be written as *magnitude* and *phase*; that is,

$$Z = |Z|e^{j\theta} \tag{4.51}$$

with

$$|Z| = \left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2} \quad \text{and} \quad \theta = \tan^{-1}\left[\frac{\omega L - \frac{1}{\omega C}}{R}\right] . \tag{4.52}$$

If we use the voltage phasor as a reference (i.e., let's take the voltage phasor to have a phase angle of zero), we write

$$V = V_m e^{j0} . \tag{4.53}$$

Current Magnitude $i|$:

Now, from (4.49), (4.51) and (4.53) the current phasor becomes

$$I = \frac{V_m}{|Z|e^{j\theta}} = \frac{V_m}{|Z|}e^{-j\theta}$$

and using (4.52)

$$I = \frac{V_m}{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}} e^{-j\theta} \ . \tag{4.54}$$

Recalling that $i = \mathcal{R}e\left\{Ie^{j\omega t}\right\}$, from (4.54) we see that magnitude of the time-domain i is

$$|i| = \frac{V_m}{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}} \ . \tag{4.55}$$

We make the following observations about equation (4.55):

(1) $|i| \to 0$ as $\omega \to 0$; and

(2) $|i| \to 0$ as $\omega \to \infty$.

This indicates that there may be a maximum for $|i|$ for some particular value of ω. In fact, from (4.55) we see that this occurs when the denominator is a *minimum* – i.e., when

$$\omega L = \frac{1}{\omega C} \Longrightarrow \omega = \frac{1}{\sqrt{LC}} \ .$$

We recognize this to be the *resonant frequency*, ω_0, of a series RLC circuit:

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{4.56}$$

When $\omega = \omega_0$, equation (4.55) indicates that $|i| = \frac{V_m}{R}$. That is, the circuit impedance is purely resistive when $\omega = \omega_0$.

From (4.55), we see that resonance occurs when the magnitudes of the inductive and capacitive reactances are equal.

The Phase Angle (ϕ) for i:

We see from the equation immediately preceding (4.54) that the phase angle for i is $-\theta$. Let's rename this as

$$\phi = -\theta \ .$$

Then, from equation (4.52)

$$\phi = \tan^{-1}\left[\frac{\frac{1}{\omega C} - \omega L}{R}\right] \tag{4.57}$$

and we see that the phase angle for the current also depends on frequency ω. Let's again consider two extremes:

(1) $\omega \rightarrow 0 \quad \Longrightarrow \quad \phi \rightarrow \tan^{-1}\left[\frac{1}{\omega RC}\right]$

and in this case the current *leads* the voltage with the phase relationship being like that of an RC circuit.

(2) $\omega \rightarrow \infty \quad \Longrightarrow \quad \phi \rightarrow \tan^{-1}\left[\frac{-\omega L}{R}\right]$

and in this case the current *lags* the voltage with the phase relationship being like that of an RL circuit.

These results should not be surprising since as

$\omega \rightarrow 0\ ,\ \ Z_c \rightarrow \frac{-j}{0}\ ,\quad$ and $\quad Z_L \rightarrow j0$

while as $\omega \rightarrow \infty\ ,\ \ Z_c \rightarrow \frac{-j}{\infty}\ ,\quad$ and $\quad Z_L \rightarrow j\infty$.

At $\omega = \omega_0$, we see from equation (4.57) that $\phi = \tan^{-1}\left[\dfrac{0}{R}\right] = 0$

Bandwidth

We define the *half-power bandwidth* of the RLC circuit as the range of frequencies (or the width of the frequency band) for which the power dissipated in R is greater than or equal to half the maximum power. We know that the average power is

$$P = |\bar{I}|^2 R \quad \text{where} \quad |\bar{I}| = \frac{I_m}{\sqrt{2}} \,. \tag{4.58}$$

Of course, *maximum* or *peak* power will be

$$P_{\text{max}} = I_m^2 R$$

and the "half-power points" occur when

$$P = \frac{P_{\text{max}}}{2} \quad \text{or} \quad |\bar{I}| = \frac{I_m}{\sqrt{2}} \,.$$

In the time domain, we know from our "resonance" considerations above that

$$i_m = \frac{V_m}{R} \,. \tag{4.59}$$

Therefore, at the *half-power points*,

$$|i| = \frac{i_m}{\sqrt{2}} = \frac{V_m}{\sqrt{2}R} \,. \tag{4.60}$$

Comparing equations (4.60) and (4.55), we observe that at the half-power points

$$\frac{V_m}{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}} = \frac{V_m}{\sqrt{2}R} \,. \tag{4.61}$$

We wish to find the frequencies for which this is true. Since the denominators in (4.61) must be equal, squaring gives

Therefore,

$$\omega =$$

and we se that, mathematically, there are 4 possible solutions (2 positive and 2 negative).

Let's take the positive ω roots so that the half-power points are associated with frequencies:

$$\omega_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} \tag{4.62}$$

and

$$\omega_2 = \frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}$$

By definition, the half-power bandwidth (BW) is given by

$$\text{BW} = \omega_2 - \omega_1 = \frac{R}{L} \tag{4.63}$$

Illustration:

From equation (4.62), it may be noted that

$$\omega_1\omega_2 = \frac{1}{LC} = \omega_0^2 \Longrightarrow \omega_0 = \sqrt{\omega_1\omega_2}\ . \tag{4.64}$$

In some circuits, the resonance may be "sharp" while in others it is "broad" – sharp resonance is often desired (for example in a radio receiver).

Illustration:

Quality Factor

A *quality factor* for the circuit is defined as

$$Q = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{\omega_0}{\text{BW}}\ . \tag{4.65}$$

A high Q implies a narrow resonance peak.

A low Q implies a broad resonance peak.

From equations (4.63), (4.64) and (4.65), the quality factor may be given in terms of the circuit parameters as

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC} = \frac{1}{R}\sqrt{\frac{L}{C}} \ . \tag{4.66}$$

From (4.66) it may be seen that for a given L, C pair, Q is inversely proportional to the resistance R; i.e., high Q requires small R (and vice versa).

Voltages Across Elements

Resistor:

Since $V_R = IR$ and at resonance, $V_R = V_m$. This also implies tthat

$$I = \frac{V_m}{R} \tag{4.67}$$

Inductor:

We know $V_L = IZ_L = j\omega LI$ and using equation (4.55) we may then write the magnitude of V_L as

$$|V_L| = \omega L|I| = \frac{\omega L V_m}{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}} \ .$$

At resonance (i.e. $\omega_0 L = 1/(\omega_0 C)$),

$$|V_L| = \frac{\omega_0 L V_m}{R} \ . \tag{4.68}$$

From equations(4.66) and (4.68), the magnitude of the inductor voltage can be written in terms of the quality factor as

$$|V_L| = QV_m \ . \tag{4.69}$$

Capacitor:

Since, in general, for the capacitor

$$V_c = \frac{I}{j\omega C} \ ,$$

using the same arguments as for the inductor (at resonance) leads to (see (4.66) also)

$$|V_c| = \frac{V_m}{\omega_0 CR} = QV_m \ . \tag{4.70}$$

We note from equations (4.69) and (4.70) that, while both V_L and V_c may be very large at resonance, they will add to give zero (recall, voltage across L and C are 180°

out of phase).

Example 1: For the following circuit, determine (i) the resonant frequency f_0; (ii) the value of R to give a quality factor of 50; and (iii) the half-power bandwidth.

(i) For resonance,

(ii) $Q = \frac{\omega_0 L}{R} \Longrightarrow R =$

(iii) First, get the half-power frequency points from equation (4.62):

$$\begin{aligned} \omega_1 &= -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} \\ \omega_2 &= \frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} \end{aligned}$$

Therefore, BW $= \omega_2 - \omega_2 =$

CHECK: From (4.63), BW $= (R/L) = 2828$ rad/s or from (4.65) BW $= (\omega_0/Q) =$ 2828 rad/s – which are the same values within rounding errors.

Example 2: Design a series RLC circuit witha resonant frequency of 1 kHz and a Q of 50 if the only available inductors have a value of 1 mH. ONE WAY:

First, find C from $\omega_0 = 1/\sqrt{LC}$: $C = 1/(L\omega_0^2) = 1/(10^{-3} \times (2\pi \times 10^3)^2) = 25.3$ μF.

Next from (4.66), $Q = 1/(\omega_0 RC) \Longrightarrow R = 1/(\omega_0 QC) = 0.129$ Ω.

4.5.2 Parallel RLC Circuits

In this section, we determine the voltage response of a parallel RLC circuit as the frequency is altered. We choose to investigate the voltage in this case because its value is the same across each of the parallel elements.

In this circuit, the impedance is obviously given by

$$Z = \frac{1}{\frac{1}{R} + \frac{1}{j\omega L} + j\omega C} \tag{4.71}$$

since $[1/(j\omega C)]^{-1} = j\omega C$. It is easy to show (VERIFY THIS) that Z may be written as

$$Z = \frac{1}{\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}} e^{j\theta} \quad \text{with} \quad \theta = \tan^{-1}\left[\left(\frac{1}{\omega L} - \omega C\right) R\right] . \tag{4.72}$$

Now, in phasor form, the voltage is

$$V = IZ = I_m \angle 0^\circ Z$$

which implies

$$V = \frac{I_m}{\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}} e^{j\theta} . \tag{4.73}$$

The frequency dependency is obvious. Clearly, the maximum voltage occurs when

$$\omega C = \frac{1}{\omega L}$$

and the frequency at which this happens is again the *resonant frequency* given by

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

which is identical to that for the RLC series circuit.

Voltage Magnitude $|V|$:

From equation (4.72), we see that $\theta = 0^\circ$ for this case so that from (4.73), the maximum voltage V_m is

$$V_m = I_m R \tag{4.74}$$

Illustration:

Voltage Phase (θ):

From (4.72), as $\omega \Longrightarrow 0$, $\theta \Longrightarrow 90°$ (the circuit "looks" inductive).

Similarly, as $\omega \Longrightarrow \infty$, $\theta \Longrightarrow -90°$ (the circuit "looks" capacitive).

As noted above, $\omega = \omega_0$ gives $\theta = 0°$.

Illustration:

Bandwidth:

As for the series circuit, we now consider the half-power bandwidth of the parallel RLC circuit. Our approach is the same as before:

First, we determine the half-power points on the $|V|$ vs. ω plot. From (4.74), we have the voltage at resonance to be $V_m = I_m R$. Recalling that power is proportional to voltage squared, half-power occurs when

$$|V| = \frac{V_m}{\sqrt{2}} = \frac{I_m R}{\sqrt{2}} \tag{4.75}$$

See the plot of voltage magnitude versus radian frequency given above. From equations (4.73) and (4.75), half-power occurs when

$$\frac{I_m R}{\sqrt{2}} = \frac{I_m}{\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}} \,. \tag{4.76}$$

As for the series case, this may be solved for (positive) ω to now give

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\frac{1}{(2RC)^2} + \frac{1}{LC}} \tag{4.77}$$

and

$$\omega_2 = \frac{1}{2RC} + \sqrt{\frac{1}{(2RC)^2} + \frac{1}{LC}}$$

Note the differences between (4.77) and the corresponding values for the series circuit found in (4.62). From (4.77), the *half-power bandwidth* is now seen to be

$$\text{BW} = \omega_2 - \omega_1 = \frac{1}{RC} \ . \tag{4.78}$$

Quality Factor (Q):

Here,

$$Q = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{1/\sqrt{LC}}{1/(RC)} = R\sqrt{\frac{C}{L}} = \omega_0 RC = \frac{R}{\omega_0 L} \ . \tag{4.79}$$

Considering (4.66) and (4.79) simultaneously, it is easily seen that

$$Q_{\text{parallel}} = \frac{1}{Q_{\text{series}}} \ .$$

This time, in contrast to the series case, a high Q is achieved via a large R.

For completeness, we note that, at resonance,

$$|I_c| = |I_L| = QI_m \ . \tag{4.80}$$

CHECK THIS by considering $V_m/|Z_c|$ and $V_m/|Z_L|$ when $\omega = \omega_0$ – for which $V_m = I_m R$.

Relating Damping to Quality Factor

The frequency response may be related to the natural response. For the parallel RLC circuit, the natural response discussed in Section 3.1.2 of these notes required that

$$\alpha = \frac{1}{2RC} \quad \text{(damping coefficient)} \ .$$

Using (4.79)

$$\alpha = \frac{\omega_2 - \omega_1}{2} = \frac{\omega_0}{2Q} \ . \tag{4.81}$$

Meanwhile, the damped frequency ω_d which (recall) is given by

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

becomes

$$\omega_d = \omega_0\sqrt{1 - \frac{1}{4Q^2}} \ . \tag{4.82}$$

Notice from equation (4.81) that the damping coefficient α is *inversely proportional to* the quality factor Q. Also, recall that the transition from *underdamped* to *overdamped* response occurs when $\omega_0^2 = \alpha^2$ (i.e., at the *critical damping* condition). Thus, from equation (4.81), we see that critical damping occurs when

$$Q = \frac{1}{2} \quad \text{(critical damping)} .$$

Similarly, from (4.81)

$$\begin{aligned} &\text{when} \quad \alpha^2 \; > \; \omega_0^2 \, , \quad Q < \frac{1}{2} \quad \text{(overdamped) and} \\ &\text{when} \quad \alpha^2 \; < \; \omega_0^2 \, , \quad Q > \frac{1}{2} \quad \text{(underdamped)} . \end{aligned}$$

<u>Example 1:</u> In the following circuit, $R = 2$ kΩ, $C = 0.25$ μF, and $L = 40$ mH. Determine (i) ω_0; (ii) Q; (iii) ω_1 , ω_2; (iv) BW; and (v) $v(\omega_0)$ if $i = 0.05 \cos \omega t$ A.

(i) $\omega_0 = 1/\sqrt{LC} =$

(ii) $Q = \omega_0 RC =$

(iii) $\omega_1 = -\dfrac{1}{2RC} + \sqrt{\dfrac{1}{(2RC)^2} + \dfrac{1}{LC}} =$

$\omega_2 = \dfrac{1}{2RC} + \sqrt{\dfrac{1}{(2RC)^2} + \dfrac{1}{LC}} =$

(iv) BW $= \omega_2 - \omega_1 = \omega_0 / Q =$

(v) At resonance, $V = V_m = I_m R =$

<u>Example 2:</u> In the following circuit, determine ω_0 and Q.

4.5.3 Filters

The ideas encountered in Sections 4.5.1 and 4.5.2 provide the basis for a brief discussion of a class of frequency-dependent circuits referred to as filters. They are so named because certain frequencies are "favoured" and certain others are "filtered out". In this section, we'll treat these circuits as voltage dividers and examine the signal across one component. There are four types of filters considered here: (1) Low-Pass; (2) High Pass; (3) Band Pass; and (4) Band Reject. Their names are indicative of the frequencies which may be obtained from them.

1. The Low-Pass Filter

A low-pass filter "allows low frequencies" but "rejects high frequencies". Consider the input voltage shown in the circuit below to be of the form

$$v_{\text{in}} = V_m \cos \omega t$$

and we wish to find v_0.

In phasor form, since $\theta_{v_{\text{in}}} = 0$,

$$I = \frac{V_m}{Z} = \frac{V_m}{R + \frac{1}{j\omega C}} = \frac{V_m \left(R + \frac{j}{\omega C}\right)}{R^2 + \left(\frac{1}{\omega C}\right)^2} \, .$$

Using magnitude and phase notation,

$$I = |I| e^{j\theta_i} = \frac{V_m e^{j\theta_i}}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \quad \text{with} \quad \theta_i = \tan^{-1}\left(\frac{1}{\omega RC}\right) \, . \tag{4.83}$$

Now, consider the phasor "output voltage" V_0 across the capacitor. Using (4.83)

$$V_0 = I Z_c = \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} e^{j\theta_i} \times \frac{1}{j\omega C}$$

Since $1/j = -j = e^{-j\pi/2}$

$$V_0 = \frac{V_m e^{j(\theta_i - \pi/2)}}{\sqrt{1 + \omega^2 R^2 C^2}} \, . \tag{4.84}$$

Taking this to the time doamin via

$$v_c(t) = \mathcal{R}e\left\{V_c e^{j\omega t}\right\} \, ,$$

$$v_0(t) = \frac{V_m}{\sqrt{1+\omega^2 R^2 C^2}} \cos(\omega t + \theta_i - \pi/2) \ . \tag{4.85}$$

Relative Magnitude Plot ($|v_0/V_m|$ vs. ω:)

Phase Plot (θ_0 vs. ω:)

From (4.83) and (4.84),

$\theta_0 = \theta_i - \frac{\pi}{2} = \left[\tan^{-1}\left(\frac{1}{\omega RC}\right) - \frac{\pi}{2}\right]$.

Note:

$\omega \Longrightarrow 0$, $\theta_0 \Longrightarrow (\frac{\pi}{2} - \frac{\pi}{2}) = 0$;

$\omega \Longrightarrow \infty$, $\theta_0 \Longrightarrow (0 - \frac{\pi}{2}) = -\frac{\pi}{2}$.

Cut-off Frequency (ω_c):

The *cut-off frequency* is defined as the frequency at which $v_0 = V_m/\sqrt{2}$.

From equation (4.85), we see that at $\omega = \omega_c$,

$$\left|\frac{v_0}{V_m}\right| = \frac{1}{\sqrt{1+\omega_c^2 R^2 C^2}} = \frac{1}{\sqrt{2}} \ . \tag{4.86}$$

Solving (4.86) for ω_c easily gives

$$\omega_c = \frac{1}{RC} \quad \text{OR} \quad f_c = \frac{1}{2\pi RC} \quad \text{in Hz.} \tag{4.87}$$

Again, from (4.85)

$$\left|\frac{v_0}{V_m}\right| = \frac{1}{\sqrt{1+\left(\frac{f}{f_c}\right)^2}} \tag{4.88}$$

since $f = \frac{\omega}{2\pi}$.

Clearly, as f becomes small, $\left|\frac{v_0}{V_m}\right| \Longrightarrow 1$.

As f becomes large, it is not too hard to convince oneself that $\left|\frac{v_0}{V_m}\right| \sim \frac{1}{f}$ where $\sim$ may be read as "varies as".

Before depicting this result, let's consider writing $\left|\frac{v_0}{V_m}\right|$ in *decibels.*

Let $A_{\mathrm{dB}} = \left|\dfrac{v_0}{V_m}\right|$ in dB;

$$A_{\mathrm{dB}} = 20\log_{10}\left|\frac{v_0}{V_m}\right| \ .$$

$\left\lvert\frac{v_0}{V_m}\right\rvert$	A_{dB}

Note that $-3\,\mathrm{dB}$ means that the output voltage drops to $V_m/\sqrt{2}$. Equivalently, the power drops to 1/2 of its maximum value.

Let's consider plotting $\left|\dfrac{v_0}{V_m}\right|$ vs. f on logarithmic axes.

Illustration:

Consider the slope of the right-hand part of the curve where f is large.

$$\text{Slope of } \left|\frac{v_0}{V_m}\right| \text{ vs. } f = \frac{\log\left|\frac{v_0}{V_m}\right|_2 - \log\left|\frac{v_0}{V_m}\right|_1}{\log f_2 - \log f_1} = m \ , \quad \text{say.}$$

Since $\log a - \log b = \log \frac{a}{b}$, the V_m cancels and

$$m = \frac{\log |v_0|_2 \,/\, |v_0|_1}{\log(f_2/f_1)} \ .$$

From our deliberations immediately following equation (4.88) and recalling that V_m is a constant (for ideal sources at least), for the high frequency part of the curve

$$|v_0|_2 \,/\, |v_0|_1 = f_1/f_2$$

to a very good approximation.

Other Terminology

The term "one *octave* change in frequency" means a change by a factor of 2. For the high-frequency part of the slope shown above, as f *increases* by a factor of 2, v_0 *decreases* by a factor of 2. That is, v_0 drops by 6 dB or

$$m = \frac{-6 \text{ dB}}{\text{octave}} \,.$$

Another "realization" of a low-pass filter is shown below:

It is easy to show that here

$$\left|\frac{v_0}{v_m}\right| = \frac{1}{\left[1 + \frac{\omega^2 L^2}{R^2}\right]^{1/2}}$$

with $\omega_c = R/L$ and $f_c = \dfrac{R}{2\pi L}$ (SHOW THIS). Then, as in (4.88), we get

$$\left|\frac{v_0}{v_m}\right| = \frac{1}{\left[1 + \left(\frac{f}{f_c}\right)^2\right]^{1/2}}$$

Example: Design a low-pass filter with $\left|\dfrac{v_0}{V_m}\right| < 0.1$ for $f > 10$ kHz.
Thus, we design for

$$\left|\frac{v_0}{V_m}\right| = \frac{1}{\left[1 + \left(\frac{f}{f_c}\right)^2\right]} = 0.1$$

when $f = 10^4$ Hz. We know

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi RC} \Rightarrow RC =$$

Now, we may choose either an R or a C to continue the design. Suppose we have a supply of 0.10 μF capacitors; then

$$R =$$

and the filter is as shown below.

2. The High-Pass Filter

As the name suggests, a high-pass filter "passes high frequencies" and "rejects low frequencies". By taking the output of our previous RC filter across the resistor, we obtain a high-pass filter. Consider the following circuit:

Here, $V_0 = IR$ in phasor form. Obviously,

$$V_0 = \frac{V_m}{R + \frac{1}{j\omega C}} R$$

and doing the algebra gives

$$V_0 = \frac{V_m \omega RC}{[1 + \omega^2 R^2 C^2]^{1/2}} e^{j\theta_0}$$

where

$$\left| \frac{v_0}{V_m} \right| = \frac{\omega RC}{[1 + \omega^2 R^2 C^2]^{1/2}} \qquad \text{with} \qquad \theta_0 = \tan^{-1} \frac{1}{\omega RC} \ .$$

Note:(i) As $\omega \to 0$, $\left| \frac{v_0}{V_m} \right| \to 0$;

(ii) As $\omega \to \infty$, $\left| \frac{v_0}{V_m} \right| \to 10$;

(iii) As $\omega \to 0$, $\theta_0 \to 90°$;

(iv) As $\omega \to \infty$, $\theta_0 \to 0°$;

Also, again defining the *cut-off* frequency as occurring when

$$\left| \frac{v_0}{V_m} \right| = \frac{1}{\sqrt{2}}$$

we get

$$\frac{\omega_c RC}{[1+\omega_c^2 R^2 C^2]^{1/2}} = \frac{1}{\sqrt{2}}$$

when $\omega = \omega_c$; therefore,

$$\omega_c = \frac{1}{RC} \ .$$

Illustrations:

A high-pass filter can also be realized using an RL circuit as shown below: It is possible to show that the cut-off frequency is given by

$$\omega_c = \frac{R}{L}$$

as in the low-pass case for these elements. It may be also easily verified that

$$\left|\frac{v_0}{V_m}\right| = \frac{f/f_c}{[1+(f/f_c)^2]^{1/2}} \ .$$

This is may be illustrated as

Examples:

1. Determine the cut-off frequency in hertz for a high-pass filter with $C = 1\ \mu\text{F}$ and R having successive values of 100 Ω, 5 kΩ, and 30 kΩ.

$$\omega_c = \frac{1}{RC} \Longrightarrow f_c = \frac{1}{2\pi RC} \quad : \quad \begin{array}{l} f_{c_1} = \\ \\ f_{c_2} = \\ \\ f_{c_3} = \end{array}$$

2. If C is replaced with a 3.5 mH inductor when $R = 5$ kΩ, what would the cut-off frequency be?

Answer: $\omega_c = \dfrac{R}{L} \Longrightarrow f_c = \dfrac{R}{2\pi L} =$

We note, in passing, that "sharper" filters may be constructed by having a series of basic filters in a circuit. For example, consider the circuit below which is referred to as a 2-pole low-pass filter:

Using our usual circuit analysis techniques, it is not hard to show that

$$\left|\frac{v_o}{V_m}\right| = \frac{1}{\left[1 + \frac{7\omega^2 L^2}{R^2} + \frac{\omega^4 L^4}{R^4}\right]^{1/2}} \qquad *$$

so that for

$\omega \to 0, \left|\dfrac{v_o}{V_m}\right| \to 1$

while for $\omega \to \infty, \left|\dfrac{v_o}{V_m}\right| \to \dfrac{1}{\omega^2}$.

The slope of the $\left|\dfrac{v_o}{V_m}\right|$ vs. f plot as shown on page 35 then becomes

$$m = \frac{-12 \text{ dB}}{\text{octave}} .$$

Furthermore, it is easy to show from $*$ that the cut-off frequency is (TRY THIS)

$$\omega_c = \frac{0.374R}{L} .$$

Band-Pass Filter

As the name suggests, *band-pass* filters pass voltages whose frequencies lie within a certain band (or range) to the output. Recall the resonance curve of a series RLC circuit. We define the bandwidth (now the filter bandwidth) as the frequency range between the half-power points:

Band-Reject Filter

If the output of the above RLC filter is configures as shown below, a *band-reject* filter with $\omega_0 = 1/\sqrt{LC}$ and a BW $= (\omega_2 - \omega_1)$ results.