5.2 Examining Circuits in the s-Domain

5.2.1 Circuit Elements

To model a circuit element in the s-domain we simply Laplace transform the voltage current equation for the element terminals in the time domain. This gives the s-domain relationship between the voltage and the current which may be modelled by an appropriate circuit. The *transformation* of a *voltage* and *current* in from the time domain results in dimensions of volt-seconds and ampere-seconds in the s-domain. *Impedance* is still measured in ohms in the s-domain. We will use the *passive sign convention* in our s-domain models. Also, we'll use V and I to mean $V(s)$ and $I(s)$, respectively.

Resistors in the s-Domain

In the time domain

Time Domain Model

$$v = iR\,.$$

Since R is a constant, in the s-domain,

s-Domain Model

$$\boxed{V = RI} \tag{5.18}$$

where $V = \mathcal{L}\{v\}$ and $I = \mathcal{L}\{i\}$.

Inductors in the s-Domain

In the time domain

Time Domain Model

$$\boxed{v = L\frac{di}{dt}} \tag{5.19}$$

(In this model I_0 is the initial current in the inductor.)

Using equation (5.8) for differentiation when Laplace transforming, equation (5.19) becomes

$$\boxed{V = L\left[sI - I_0\right] = sLI - LI_0} \tag{5.20}$$

where $I_0 = i(0^-)$. Of course, if there is no initial current, $I_0 = 0$. We note that equation (5.20) may also be written as

$$\boxed{I = \frac{V}{sL} + \frac{I_0}{s}} \tag{5.21}$$

While the inductor may be modelled in various equivalent ways in the s-domain, the last two equations immediately suggest two of these:

<u>Series Model – Equation (5.20):</u>

(sL) is the s-domain impedance.

LI_0 is like a constant voltage whose value depends on the initial conditions.

<u>Parallel Model – Equation (5.21):</u>

This time, I_0/s is like an independent (constant)current source depending on initial conditions.

Of course, if $i(0^-) = 0$, both of the above reduce to:

–i.e. the inductor transforms to an impedance (sL).

Capacitors in the s-Domain

In the time domain

<u>Time Domain Model</u>

$$\boxed{i = C\frac{dv}{dt}} \tag{5.22}$$

(In this model V_0 allows for the possibility of an initial voltage across the capacitor.)

Converting (5.22) via Laplace transformation gives

$$\boxed{I = C\left[sV - V_0\right] = sCV - CV_0} \tag{5.23}$$

where $V_0 = v(0^-)$. Of course, if there is no initial voltage, $V_0 = 0$. We note that equation (5.23) may also be written as

$$\boxed{V = \left(\frac{1}{sC} I\right) + \frac{V_0}{s}} \tag{5.24}$$

While the capacitor may be modelled in various equivalent ways in the s-domain, the last two equations immediatly suggest two of these:

Series Model – Equation (5.24):

$1/(sC)$ is the s-domain impedance.
V_0/s is like a constant voltage whose value depends on the initial conditions.

Parallel Model – Equation (5.23):

This time, CV_0 is like an independent (constant)current source depending on initial conditions.

Of course, if $v(0^-) = 0$, both of the above reduce to:

–i.e. the capacitor transforms to an impedance ($1/(sC)$).

5.2.2 s-Domain Circuit Analysis

1. General

In the s-domain, if no energy is stored in the inductor or capacitor, the relationship between V and I for each passive element of impedance Z is still

$$\boxed{V = IZ} \tag{5.25}$$

In this domain, $Z_R = R$, $Z_L = sL$ and $Z_C = 1/(sC)$.
Techniques involving Kirchoff's Laws (KVL and KCL), Node-Voltage, Mesh-Current, Delta-Wye Transformation, Thévenin, etc., etc. still hold! If there is initially stored energy, equation (5.25) may be modified by adding the appropriate independent sources in series or parallel with the element impedances as depicted in the previous subsection.

2. Application 1 – Natural Response of an RC Circuit

In the circuit shown below on the left, the capacitor has an initial voltage of V_0, and we wish to find the time-domain expressions for i and v.

Method 1:

Time Domain **s-Domain**

Using KVL on the s-domain circuit, we get

$$\begin{aligned} -\frac{V_0}{s} + IZ_c + IZ_R &= 0 \\ \Longrightarrow \frac{V_0}{s} &= IZ_c + IZ_R \\ \Longrightarrow \frac{V_0}{s} &= \frac{I}{sC} + IR\,. \end{aligned}$$

From this

$$I = \frac{CV_0}{1 + RCs}\,.$$

Dividing by RC in the numerator and denominator on the right puts I into a recognizable form for inverse Laplace transformation:

$$I = \frac{V_0/R}{s + \frac{1}{RC}}\,.$$

The form is obviously $\dfrac{K}{s+a}$ and

$$\boxed{\mathcal{L}^{-1}\{I\} = i = \frac{V_0}{R} e^{-t/RC} u(t)} \tag{5.26}$$

Then,

$$\boxed{v = iR = V_0 e^{-t/RC} u(t)} \tag{5.27}$$

[Remembering that $u(t) = 1$ for $t \geq 0^+$, this is the same as we had before.]

Method 2:

We may also find v before finding i by employing the "parallel model" for the capacitor as follows:

Redraw the original time-domain circuit as

Using node-voltage at A:

$$\begin{aligned}
-CV_0 + \frac{V}{1/sC} + \frac{V}{R} &= 0 \\
\Longrightarrow V = \frac{CV_0}{sC + 1/R} &= \frac{V_0}{s + 1/RC} \; .
\end{aligned}$$

Then,

$$v = \mathcal{L}^{-1}\{V\} = V_0 e^{-t/RC} u(t)$$

as in equation (5.27).

Application 2 – Step Response of a Parallel RLC Circuit

Assumed that there is no energy stored in the circuit shown below when the switch is opened at time $t = 0$. We wish to find $i_L(t)$.

Note that $I_{\text{source}} = i_{\text{dc}} u(t)$ and $\mathcal{L}\{i_{\text{dc}} u(t)\} = I_{\text{dc}}/s$.

Because there is no initial stored energy (i.e. $i_L(0^-) = 0$ and $v_C(0^-) = 0$, the general form of the s-domain circuit is

Using KCL at the top node,

$$-\frac{I_{\text{dc}}}{s} + I_C + I_R + I_L = 0 \; .$$

This implies

$$\frac{I_{\text{dc}}}{s} = sCV + \frac{V}{R} + \frac{V}{sL}$$

from which

$$V = \frac{\frac{I_{\rm dc}}{s}}{\left[sC + \frac{1}{R} + \frac{1}{sL}\right]} \times \frac{s/C}{s/C}$$

$$V = \frac{\frac{I_{\rm dc}}{C}}{\left[s^2 + \frac{s}{RC} + \frac{1}{LC}\right]} \qquad \text{(A)} .$$

However, $I_L = V/sL$ so that from (A) we have

$$I_L = \frac{\frac{I_{\rm dc}}{LC}}{s\left[s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right)\right]} \qquad \text{(B)} .$$

Substituting the values of R, L and C into (B) results in

$$I_L = \frac{384 \times 10^5}{s(s^2 + 64,000s + 16 \times 10^8)} .$$

Factoring the denominator allows us to expand I_L using partial fractions:

$$\begin{aligned} I_L &= \frac{384 \times 10^5}{s(s + 32,000 - j24000)(s + 32,000 + j24000)} \\ &= \frac{K_1}{s} + \frac{K_2}{(s + 32,000 - j24000)} + \frac{K_2^*}{(s + 32,000 + j24000)} . \end{aligned}$$

Now,

$$K_1 = I_L|_{s=0} = \frac{384 \times 10^5}{(32,000^2 + 24000^2)} = 0.024 .$$

Also,

$$K_2 = I_L(s + 32000 - j24000)|_{s=-32,000+j24000} = \frac{384 \times 10^5}{(-32,000{+}j24000)(j48,000)} = 0.020\angle 126.87^\circ$$

which immediately gives $K_2^* = 0.020\angle - 126.87^\circ$. Now,

$$\mathcal{L}^{-1}\{K_1/s\} = \mathcal{L}^{-1}\{0.024/s\} = 0.024u(t) .$$

On page 9 of this unit, we saw that the complex conjugate pair transforms to

$$2|K|e^{-\alpha t}\cos(\beta t + \theta)u(t)$$

where here

$$|K| = |K_2| = 0.020 \quad ; \quad \alpha = 32,000 \quad ; \quad \beta = 24,000 \quad \text{and} \quad \theta = 126.87^\circ .$$

Therefore,

$$i_L = \left[0.024 + 0.040e^{-32,000t}\cos(24,000t + 126.87^\circ)\right]u(t) \quad \text{A} .$$

Again, note that the multiplier $u(t)$ accounts for $t \geq 0$. Note that $i_L(0) = 0$ and $i_L(\infty) = 0.024$ A, as should be the case.

Application 3 – Multiple Meshes (Transients) – Step Response Example

While multiple node-voltage or mesh-current analysis leads to simultaneous differential equations in the *time* domain, Laplace transforms allow us to replace these equations with simultaneous algebraic systems in the s-domain. This is illustrated with an example below. Also, see the example on pages 593-595 of the text.

Drill Exercise 13.5, page 595 of text: In the following circuit, the dc current and voltage sources are applied at the same time. There is no initially stored energy in any of the circuit components. (a) Derive the s-domain expressions for V_1 and V_2. (b) For $t > 0$, derive the time domain expressions for v_1 and v_2. (c) Determine $v_1(0^+)$ and $v_2(0^+)$. (d) Find the steady-state values of v_1 and v_2.

(a) First, represent the circuit in the s-domain.

Next, apply the node-voltage technique to nodes 1 and 2:

$$\underline{\text{Node 1:}} \quad \frac{V_1 - V_2}{s} + sV_1 - \frac{5}{s} = 0 \quad ; \underline{\text{Node 2:}} \quad \frac{V_2 - V_1}{s} + \frac{V_2}{3} + \frac{V_2 - (15/s)}{15} = 0$$

Solving for V_1 and V_2, we get

$$V_1 = \frac{5(s+3)}{s(s^2 + 2.5s + 1)} \quad \text{and} \quad V_2 = \frac{2.5(s^2+6)}{s(s^2 + 2.5s + 1)} .$$

(b) Partial fraction expansion gives

$$V_1 = \frac{15}{s} - \frac{50/3}{s+0.5} + \frac{5/3}{s+2} \quad \text{and} \quad V_2 = \frac{15}{s} - \frac{125/6}{s+0.5} + \frac{25/3}{s+2}$$

Now, the $1/s$ and $1/(s+a)$ forms are readily recognizable from the transform tables so that

$$\begin{aligned} v_1(t) &= \mathcal{L}^{-1}\{V_1\} = \left[15 - \frac{50}{3}e^{-0.5t} + \frac{5}{3}e^{-2t}\right] u(t)\ V \\ v_2(t) &= \mathcal{L}^{-1}\{V_2\} = \left[15 - \frac{125}{6}e^{-0.5t} + \frac{25}{3}e^{-2t}\right] u(t)\ V \end{aligned}$$

(c) From part (b), $v_1(0^+) = 15 - \frac{50}{3} + \frac{5}{3} = 0$; $v_2(0^+) = 15 - \frac{125}{6} + \frac{25}{3} = 2.5$.

[Note, from the **initial value theorem**, $v_1(0^+) = \lim_{s\to\infty} sV_1(s) = 15 - \frac{50}{3} + \frac{5}{3} = 0$ and similarly for $v_2(0^+)$.]

(d) Here, again, we may find $v_1(\infty)$ and $v_2(\infty)$ from part (b) or we may use the **final value theorem**:

$$
\begin{aligned}
v_1(\infty) &= \lim_{s\to 0} sV_1(s) = 15 - 0 + 0 = 15 \text{ V} \\
v_2(\infty) &= \lim_{s\to 0} sV_2(s) = 15 - 0 + 0 = 15 \text{ V}
\end{aligned}
$$

which is what we would get using the results in part (b) also.

Application 4 – Thévenin Equivalent in the s-domain

Drill Exercise 13.6, page 598 of text: (a) Given the following circuit, find the s-domain Thévenin equivalent with respect to the terminals "a" and "b". There is no initial charge on the capacitor.

First, sketch the s-domain equivalent to the left of the a-b terminals.

With no load across the a-b terminals, there is no current in the 5 Ω resistor (this is the "tricky" observation here) so that

$$V_x = \frac{20}{s} \quad \ldots \ (1)$$

Now, determine V_{Th} by applying the node-voltage rule at node 1:

Using equation (1) and simplifying gives

$$V_{\mathrm{Th}} = \frac{20(s+2.4)}{s(s+2)} \quad \ldots \ (2)$$

Next, we seek Z_{Th}: The Thévenin equivalent impedance (in the s-domain) may be found by applying a test source across the a-b terminals while shorting the independent power supply.:

Apply node-voltage at node 2 while noting

$$V_x = 5I_T \quad \ldots\ (3)$$

Thus, the node-voltage technique, incorporating test voltage V_T, gives:

$$-I_T + \frac{V_T - V_x}{2/s} + \frac{V_T - 0.2V_x - V_x}{1} = 0 \quad \ldots\ (4)$$

Simplifying (4) gives

$$Z_{\text{Th}} = \frac{V_T}{I_T} = \frac{5(s+2.8)}{(s+2)} \quad \ldots\ (5)$$

The Thévenin equivalent circuit is shown below (to the left of terminals a-b). The s-domain load is also shown.

(b) Find I_{ab} in the s-domain for the given load.

Clearly,

$$I_{\text{ab}} = \frac{V_{\text{Th}}}{Z_{\text{Th}} + Z_L} \ \text{ where } \ Z_L = 2 + s\ .$$

Using equations (2) and (5),

$$I_{\text{ab}} = \frac{20(s+2.4)}{s[(s+6)(s+3)]}\ .$$

For practice using the Laplace transform tables, you should find the time-domain current corresponding to the last expression.