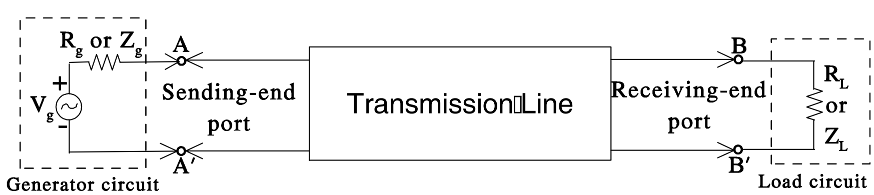
Unit 4

Applications

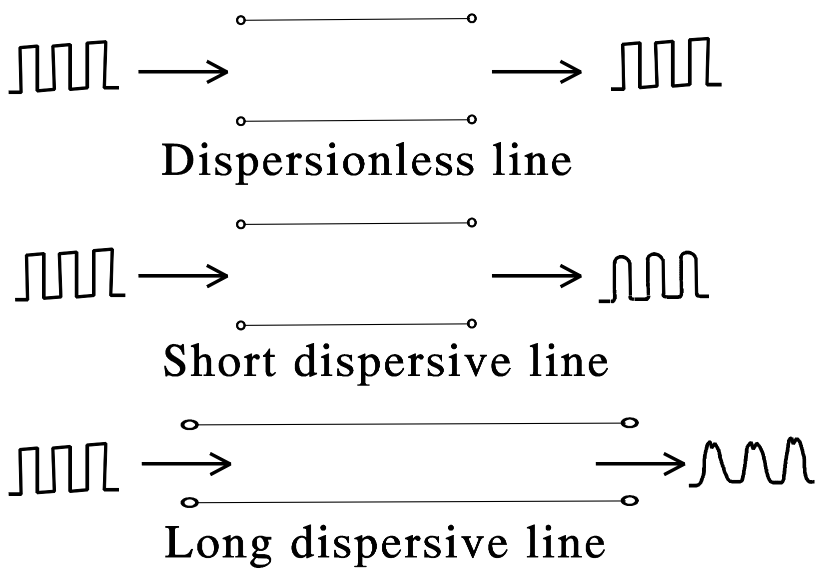
4.1 Transverse Electromagnetic (TEM) Transmission Lines

In this unit, we shall consider structures used to guide electromagnetic signals. Such a structure is referred to as a *transmission line* (TL). Examples include telephone wires, coaxial cables carrying audio and video signals, optical fibres for carrying data at very high rates, etc..

Fundamentally, a transmission line is a two-port network, with each port consisting of two terminals as shown with V_g being the generator voltage and R_g the generator resistance (in the case of a-c transmissions, we may consider the phasor voltage and the impedance Z_g and load impedance Z_L):

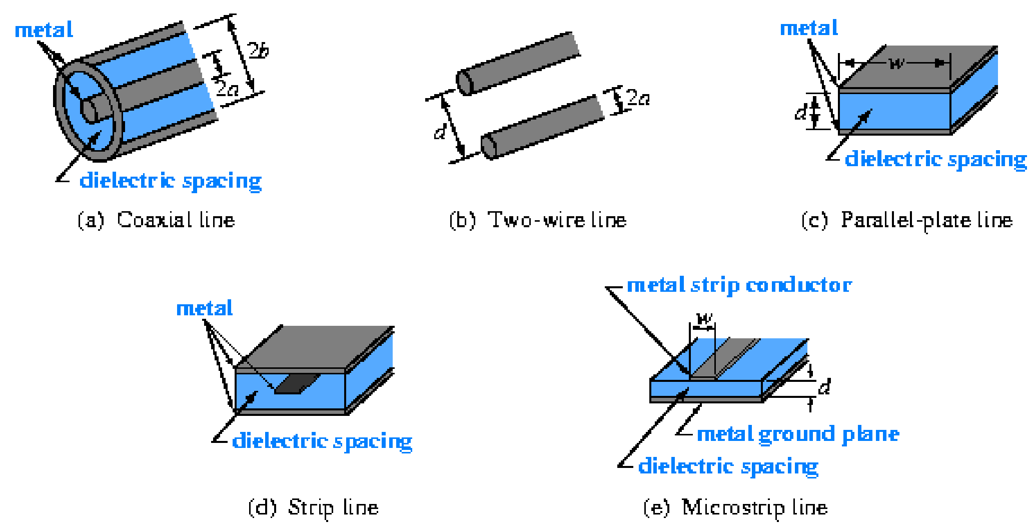


Whether or not the TL actually significantly affects the transmission of the signal from the source to the load is highly dependent on the ratio of the line length ℓ to the transmission wavelength λ. As a rule of thumb, TL effects may become important when $\ell/\lambda \gtrsim 0.01$. Then, it may be necessary to account for *phase shift* associated with time delay and the presence of *reflected signals* bounced back from the load toward the generator. Additionally, power dissipation and TL *dispersion* effects may also have to be considered. The latter arise because the frequencies constituting a signal may not all propagate at the same velocity. [See diagram].



These are very real problems that become pronounced and debilitating as frequency increases. For example at 10 GHz, a signal, which in air has a wavelength of 3 cm, has a wavelength of about 1 cm in a semiconductor material. Thus, connections whose lengths are on the order of millimetres could be critically important for, for example, semiconductor devices on an IC chip. The characteristics of such TL have to be therefore incorporated into the overall circuit design.

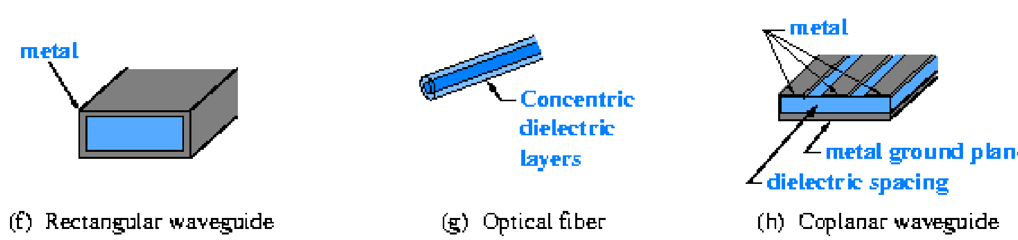
As intimated, there are many kinds of TL, but they may be classified into two basic types: (1) The transverse electromagnetic (TEM) TL in which both the electric and magnetic fields are *transverse* to the direction of propagation and (2) higher-order TL in which at least one of the E or H fields has a significant component along the direction of propagation. We cite the e-m waveguide as an example of the second class – this structure is often used at GHz frequencies and is unsurpassed for low-loss transmission of high power at these frequencies. Illustrations of the two groups of transmission lines are shown below. In this course, we shall consider only the TEM lines.



(a) Coaxial line (b) Two-wire line (c) Parallel-plate line

(d) Strip line (e) Microstrip line

TEM Transmission Lines

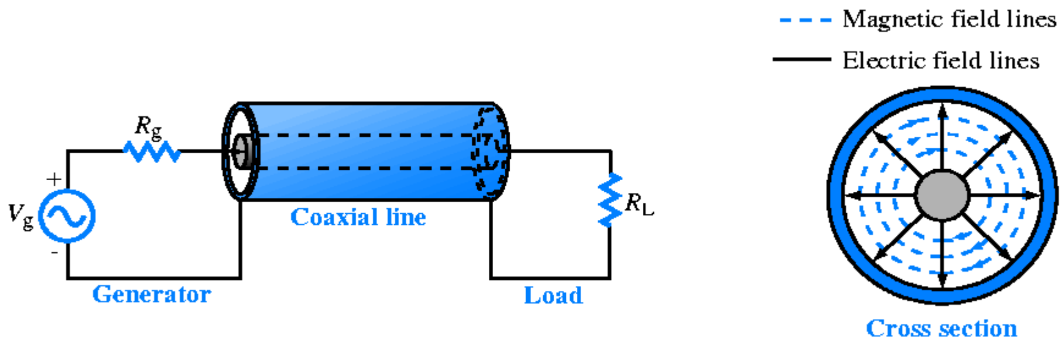


(f) Rectangular waveguide (g) Optical fiber (h) Coplanar waveguide

Higher Order Transmission Lines

4.1.1 The Telegrapher's Equations and Lossless Transmission Lines

What follows is relevant to the simple two-wire TL or the coaxial line depicted above. In particular, we note that $\vec{E}$ and $\vec{H}$ fields are oriented as shown.



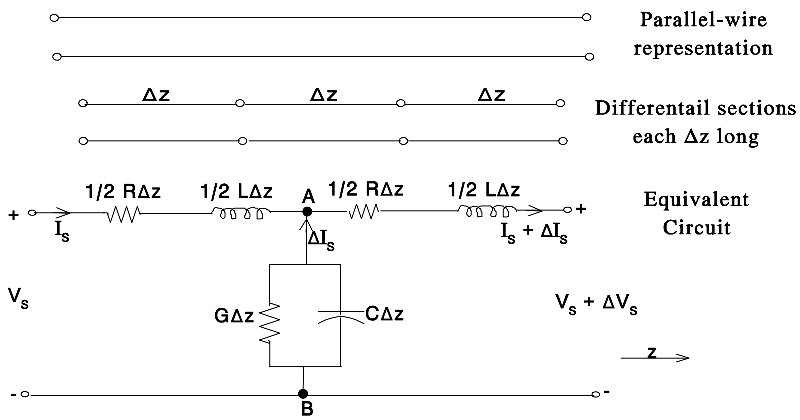
The coaxial line has a useful bandwidth up to several 100's of megahertz while a two-wire twisted-pair line may have a bandwidth up several 10's of megahertz depending on wire guage, insulation, signal levels and the length of the line.

For purposes of developing the theory we shall concentrate on the coaxial line. We could, in fact, analyze the TL in terms of the $\vec{E}$ and $\vec{H}$ fields and Maxwell's equations with appropriate boundary conditions, but it is more commonly done using voltage (V) and current (I). The TL may be depicted (electrically) by a parallel wire configuration consisting of several "lumped" elements: i.e., each identical short length, Δz, of wire will consist of 4 basic elements which we shall refer to from now on as *transmission line parameters*:

- R: The combined *resistance per unit length* of both conductors in Ω=m.
- L: The combined *inductance per unit length* of both conductors in H=m.
- G: The *conductance per unit length* of the insulation ℧=m.
- C: The *capacitance per unit length* of the two conductors in F=m.

We shall carry out a time-harmonic analysis of such a TL and use V_s and I_s as the voltage and current phasors, respectively. We may choose from among various lumped-parameter models, which will give the same results in the limit as Δz shrinks to differential size (i.e. as $\Delta z \to dz$).

Illustration:



The voltage equation for this section is , clearly,

from which

Assuming that $\Delta I_s \ll I_s$ and taking the limit $(\Delta z \to 0)$,

$$\boxed{\lim_{\Delta z \to 0} \left(\frac{\Delta V_s}{\Delta z} \right) = \frac{dV_s}{dz} = -(R + j\omega L) I_s} \,. \tag{4.1}$$

For the current, assuming that the voltage drop across the parallel section from A to B is $\approx V_s + \frac{1}{2}\Delta V_s$, noting that the admittance, Y, is given by

and also noting that ΔI_s opposes $(V_s + \frac{1}{2}\Delta V_s)$ we have

Again, assuming that $\Delta V_s \ll V_s$,

and

$$\boxed{\lim_{\Delta z \to 0} \left(\frac{\Delta I_s}{\Delta z} \right) = \frac{dI_s}{dz} = -(G + j\omega C) V_s} \,. \tag{4.2}$$

We note that in the time domain, (4.1) and (4.2) become

(4.3)

and

(4.4)

Equations (4.3) and (4.4), or their equivalent phasor forms (4.1) and (4.2), respectively, are referred to as the *telegrapher's equations* and describe the current and voltage relationships on a transmission line. See from the text pp 438-439 the analogy between equation (4.1) and Faraday's law and equation (4.2) and Ampère's law.

In what follows, we assume that $(R + j\omega L)$ and $(G + j\omega C)$ are constant for the TL under consideration – i.e., the lines are *uniform*, not "tapered", for example.

We notice that the telegrapher's equations are coupled (in V_s and I_s) – this is not surprising as we have already seen from Maxwell's equations that the $\vec{E}$ and $\vec{H}$ field equations are coupled. We proceed as usual to do the "uncoupling":
Take the derivative of (4.1) w.r.t. z to get

(4.5)

Comparing (4.5) and (4.2),

which may be written as

$$\boxed{\frac{d^2V_s}{dz^2} - \gamma^2 V_s = 0} \tag{4.6}$$

where

or

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta \tag{4.7}$$

In equation (4.7), note the analogy with the earlier propagation constant ($\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$) that we had in Section 3-4. Of course, now α and β will be in terms of the transmission line parameters.

From previous experience we know that the solution to (4.6) is

$$V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \; , \tag{4.8}$$

which is the same "kind of" solution as we had for the plane wave E-field. While it isn't critical to the argument, we'll assume that the constants V_0^+ and V_0^- are real.

The two terms on the right hand side of (4.8) are linearly independent solutions as before and we'll choose the first member for our investigations. In view of (4.7), equation (4.8) may be written for the positively travelling (i.e. the $+\hat{z}$ direction) voltage "wave"

$$V_s(z) = V_0^+ e^{-\alpha z} e^{-j\beta z} \ , \tag{4.9}$$

which is a wave being attentuated in the direction of travel and which is also being phase shifted.

Having obtained an expression for the voltage, we may now, proceed to find an expression for I_s from equation (4.1):

or

$$I_s(z) = \frac{V_0^+}{Z_0} e^{-\alpha z} e^{-j\beta z} \tag{4.10}$$

where we have defined

or

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = Z_{0_m} e^{j\theta_{Z_0}} \tag{4.11}$$

Here, Z_{0_m} and θ_{Z_0} are the magnitude and phase, respectively, of the quantity Z_0 which is referred to as the *characteristic impedance* of the transmission line. The unit of Z_0 is the ohm. We shall see that THIS IS AN EXTREMELY IMPORTANT PARAMETER. [For anyone who hasn't noticed, the next time you're in Radio Shack (or some other electronics shop) see that the flat 2-wire "transmission line" for connecting a TV to an antenna is typically $300\,\Omega$ while the cable TV coaxial connecting wire is listed as $75\,\Omega$.]

Taking (4.9) and (4.10) to the time domain we obtain (since V_0^+ has been chosen to be real)

$$\underset{\sim}{V}(z) = V_0^+ e^{-\alpha z} \cos(\omega t - \beta z) \tag{4.12}$$

and

$$\underset{\sim}{I}(z) = \frac{V_0^+}{Z_{0_m}} e^{-\alpha z} \cos(\omega t - \beta z - \theta_{Z_0}) \tag{4.13}$$

where we have used (4.11) to write (4.13). [Of course, $\underset{\sim}{V}(z) \equiv V(z,t)$ and $\underset{\sim}{I}(z) \equiv I(z,t)$]. From (4.12) and (4.13), we see that $\underset{\sim}{V}$ and $\underset{\sim}{I}$ are waves whose *phase velocity*, v_p, is found, as before, by setting

and taking derivatives (in either case) to get

which implies

$$v_p = \frac{\partial z}{\partial t} = \frac{\omega}{\beta} \tag{4.14}$$

Also, the wavelength $(z = \lambda)$ must be given by $\beta\lambda = 2\pi$ or

$$\beta = \frac{2\pi}{\lambda}. \tag{4.15}$$

This further gives

or

$$v_p = f\lambda\ . \tag{4.16}$$

In passing, we note the analogies between plane wave quantities $\vec{E}$, $\vec{H}$, μ, ϵ, σ and η and their respective transmission counterparts V, I, L, C, G, Z_0.

Special Case: LOSSLESS LINES:

A TL is termed *lossless* if $R = G = 0$. In this case, equation (4.7) for the propagation constant becomes

which implies

$$\alpha = 0 \qquad ; \qquad \beta = \omega\sqrt{LC} \tag{4.17}$$

The phase velocity follows immediately as

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \tag{4.18}$$

and the characteristic impedance from (4.11) becomes

$$Z_0 = \sqrt{\frac{L}{C}} \tag{4.19}$$

An important quantity associated with a TL is what manufacturers list as a *velocity factor* (VF), given in percent. It indicates the speed of a wave on the TL as a percentage of the speed of light in a vacuum. That is

$$\text{VF} = \frac{v_p}{c} \times 100\% \tag{4.20}$$

Example: Lossless Coaxial Transmission Line

A customer gives the following specifications for a lossless high frequency coaxial TL to a manufacturer: The characteristic impedance is to be $50\,\Omega$, the dielectric is nonmagnetic, and the velocity factor is to be 65%. Determine how the manufacturer is to specify the relative permittivity of the dielectric and the ratio b/a for the required line. Without proof, we state the capacitance and inductance for such a line to be (see text, page 442).

4.1.2 Reflection and Transmission

CASE 1: Lossless Line Connected to a Lossy Line

Reflection and Transmission Coefficients

Let's consider initially the case of a voltage (or current) wave travelling along a lossless transmission line of characteristic impedance Z_{0_1} which encounters a second line with $Z_0 = Z_{0_2}$ (not necessarily lossless). We shall take $z = 0$ (i.e. the z-axis origin) at the boundary between the two transmission lines, $z < 0$ on the first line and $z > 0$ on the second line as depicted:

As with plane waves encountering a boundary, we must now consider the incident voltage, V_i, the reflected voltage, V_R, and the transmitted voltage, V_T, and the corresponding current waves. We shall symbolize the reflection and transmission coefficients by Γ and τ, respectively. The phasor equations for V and I are thus

Note that only β appears in the exponentials associated with the first line since that line is lossless (i.e. $R_1 = G_1 = 0$ and from equation (4.7) $\gamma_1 = \alpha_1 + j\beta_1 = 0 + j\beta_1$).

Again from (4.7), for the lossless line, $j\beta_1 = j\omega\sqrt{L_1 C_1}$ or $\beta_1 = \omega\sqrt{L_1 C_1}$. Also, from (4.11), $Z_{0_1} = \sqrt{L_1/C_1}$. Finally, we note that equation (4.21.e) follows from using V_R in equation (4.1). TRY THIS.

Now, across the boundary, the voltage is continuous (recall the E-field aguments for plane waves). Therefore,

and from (4.21.a)-(4.21.c) with $z = 0$ (i.e. at the boundary)

which gives

$$\boxed{1 + \Gamma = \tau} \tag{4.22}$$

Also, I is continuous across the boundary (recall the H-field arguments for plane waves), so that from (4.21.d)-(4.21.f), with $z = 0$,

$$\boxed{\frac{1}{Z_{0_1}} - \frac{\Gamma}{Z_{0_1}} = \frac{\tau}{Z_{0_2}}} \tag{4.23}$$

Clearly, (4.22) and (4.23) are completely analogous to equations (3.84) and (3.85) of Section 3.6 (with Z's replacing η's). Therefore,

$$\boxed{\Gamma = \frac{Z_{0_2} - Z_{0_1}}{Z_{0_2} + Z_{0_1}}} \tag{4.24}$$

and

$$\boxed{\tau = \frac{2Z_{0_2}}{Z_{0_2} + Z_{0_1}}} \tag{4.25}$$

[Compare (4.24) and (4.25) with their plane-wave counterparts in equations (3.86) and (3.87) of Section 3-6.] Note that if the lines have the same characteristic impedances, $\Gamma = 0$ and $\tau = 1$.

Standing Wave Ratio

From equations (4.21.a) -(4.21.e) we notice that the total voltage and current at position z on the first TL may be written

$$V_1 = V_i + V_R = V_0^+ \left[e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z} \right] \tag{4.26}$$

$$I_1 = I_i + I_R = \frac{V_0^+}{Z_{0_1}} \left[e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z} \right] \tag{4.27}$$

These equations are completely analogous to equations (3.97) and (3.98) of Section 3.6. It is thus easy to show, as was done for equation (3.102) of that section, that the SWR is still given by

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} \; . \tag{4.28}$$

Additionally, by the same token as before, the power density ratio, p_r, is still related to the SWR by

$$p_r = \left(\frac{s - 1}{s + 1} \right)^2 \; . \tag{4.29}$$

<u>Input Impedance</u>

Consider next that we wish to determine the impedance, Z_{in}, as "viewed" at a position $z = -\ell$ on the first line:

Of course,

$$Z_{\text{in}} = \frac{V_{\text{in}}}{I_{\text{in}}} \tag{4.30}$$

so that the problem reduces to establishing the voltage and current expressions at $z = -\ell$.

From equations (4.26) and (4.27) with $z = -\ell$,

$$V_{\text{in}} = V_0^+ \left[e^{j\beta_1 \ell} + \Gamma e^{-j\beta_1 \ell} \right] \tag{4.31}$$

and

$$I_{\text{in}} = \frac{V_0^+}{Z_{0_1}} \left[e^{j\beta_1 \ell} - \Gamma e^{-j\beta_1 \ell} \right] \tag{4.32}$$

with Γ given by (4.24). Then,

$$Z_{\text{in}} = \frac{V_{\text{in}}}{I_{\text{in}}} = Z_{0_1} \left[\frac{e^{j\beta_1 \ell} + \left(\frac{Z_{0_2} - Z_{0_1}}{Z_{0_2} + Z_{0_1}} \right) e^{-j\beta_1 \ell}}{e^{j\beta_1 \ell} - \left(\frac{Z_{0_2} - Z_{0_1}}{Z_{0_2} + Z_{0_1}} \right) e^{-j\beta_1 \ell}} \right]$$

$$Z_{\text{in}} = Z_{0_1}\left[\frac{Z_{0_2} + jZ_{0_1}\tan\beta_1\ell}{Z_{0_1} + jZ_{0_2}\tan\beta_1\ell}\right] \tag{4.33}$$

Equation (4.33) is the input impedance as seen at $z = -\ell$.

Further, let's suppose that the second transmisison line is replaced with an equivalent "lumped" impedance, Z_L, at $z = 0$. Since there is now only one transmission line, we'll let its characteristic impedance be Z_0 and $\beta_1 = \beta$.

It is easy to show that (4.24), (4.28) and (4.33) may be appropriately altered by simply replacing Z_{0_2} with Z_L. At $z = 0$, equations (4.31) and (4.32) become

It is readily shown (DO THIS) from these expressions that

$$\Gamma = \frac{Z_L - Z_0}{Z_L - Z_0} \tag{4.34}$$

Furthermore, noting the similarity between equation (4.26) of this section and (3.97) of Section 3.6, it is straightforward to determine that the standing wave ratio in the present case is given by

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} \tag{4.35}$$

Also, the input impedance at $z = -\ell$ is easily shown to be

$$Z_{\text{in}} = Z_0\left[\frac{Z_L + jZ_0\tan\beta\ell}{Z_0 + jZ_L\tan\beta\ell}\right] \tag{4.36}$$

CASE 2: Lossy Line Connected to a Lumped Impedance, Z_L

We now consider any necessary modifications to the above results if the transmission line connected to Z_L is lossy.

Now we have to consider the general propagation constant $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$ and the characteristic line impedance $Z_0 = \sqrt{\dfrac{R + j\omega L}{G + j\omega C}}$. The incident and reflected voltages and currents are given by equations (4.21.a,b,d,e) with β_1 being replaced by γ because the line is lossy. At the load, as seen in the previous case,

$$V_L = [V_i + V_R]_{z=0} = V_0^+(1 + \Gamma) \text{ and } I_L = [I_i + I_R]_{z=0} = \frac{V_0^+}{Z_0}(1 - \Gamma)$$

from which, as before (see equation (4.34),

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{4.37}$$

Therefore, the total voltage at $z = -\ell$ is given by (see (4.21.a,b) with β_1 replaced by γ)

$$V_{\text{in}} = [V_i + V_R]_{z=-\ell} = V_0^+(e^{\gamma\ell} + \Gamma e^{-\gamma\ell})$$

Therefore,

$$V_{\text{in}} = \frac{2V_0^+}{Z_L + Z_0}\left[Z_L \cosh\gamma\ell + Z_0 \sinh\gamma\ell\right] \tag{4.38}$$

The current $I_{\text{in}} = [I_i + I_R]_{z=-\ell}$ can be determined in an analogous fashion and shown to be (DO THIS)

$$I_{\text{in}} = \frac{2V_0^+}{Z_0(Z_L + Z_0)}\left[Z_L \sinh\gamma\ell + Z_0 \cosh\gamma\ell\right] \tag{4.39}$$

Therefore,

$$Z_{\text{in}} = \frac{V_{\text{in}}}{I_{\text{in}}} =$$

or

$$Z_{\text{in}} = Z_0 \left[\frac{Z_L + Z_0 \tanh \gamma\ell}{Z_0 + Z_L \tanh \gamma\ell} \right] \tag{4.40}$$

The similarity in form with (4.33) where $\gamma = j\beta_1$ is striking.

Next, we note that by definition

$$\tanh(\alpha + j\beta)\ell =$$

and dividing each term of the numerator and denominator by $e^{(\alpha+j\beta)\ell}$ we get

$$\tanh(\alpha + j\beta)\ell = \frac{1 - e^{-2\alpha\ell}e^{-2j\beta\ell}}{1 + e^{-2\alpha\ell}e^{-2j\beta\ell}}$$

It may be observed that if the line is lossy (i.e. $\alpha \neq 0$), then for large ℓ this tanh $\to 1$. Checking equation (4.40), we see that for this case (i.e. lossy line and ℓ large),

$$Z_{\text{in}} \to Z_0 \left[\frac{Z_L + Z_0(1)}{Z_0 + Z_L(1)} \right] \to Z_0 \quad .$$

This is a very important result. It says that for high-loss lines, $Z_{\text{in}} \to Z_0$, no matter the value of Z_L. That is, a source (perhaps a transmitter of some kind) "sees" essentially only the line impedance in this case. What's happening physically in this situation is that V_R and I_R are so attenuated, they become very small (insignificant) compared to V_{in}. Thus, while equation (4.35) for the SWR remains valid if s is measured near the "load", when measured at the input of the long, lossy line, $s \to 1$.

4.2 Special Cases of the Lossless Line

4.2.1 The Short-Circuit Line

Through the proper choice of the length of a short-circuit line, it is possible to make substitutes for capacitors or inductors of any desired reactance. This practice is common in the design of microwave circuits and high-speed intgrated circuits, because in these applications it is easier to make a shorted transmission line than to make an actual capacitor or inductor.

Consider such a lossless shorted TL of length ℓ as shown:

In this case, the load impedance is zero ($Z_L = 0$) so that from equations (4.34), (4.35) and (4.29),

i.e., all incident power is being reflected. From equation (4.36), we have for the case at hand

$$Z_{\text{in}} =$$

Since Z_0 and $\beta = \omega\sqrt{LC}$ are both real, the input impedance is now purely reactive – i.e. either capacitive or inductive.

Case 1: $0 < \beta\ell < \dfrac{\pi}{2}$

For this case, $0 < \tan\beta\ell < \infty$. Writing β as $\frac{2\pi}{\lambda}$, we have

and the input impedance at $z = -\ell$ is a positive quantity of the form jX – i.e. $X = \omega L_{\text{eq}}$ where L_{eq} is an equivalent inductance. The equivalent circuit is

and we have

We notice that L_{eq} changes with frequency. Because of the range of $\tan\beta\ell$, any value of L_{eq} may be realized.

Case 2: $\frac{\pi}{2} < \beta\ell < \pi$

For this case, $-\infty < \tan \beta\ell < 0$. Again, writing β as $\frac{2\pi}{\lambda}$, we have

and the input impedance at $z = -\ell$ is a *negative* quantity of the form jX – i.e. $X = -1/(\omega C_{\text{eq}})$ where C_{eq} is an equivalent capacitance. The equivalent circuit is

and we have

We notice that C_{eq} changes with frequency. Because of the range of $\tan \beta\ell$, any value of C_{eq} may be realized.

Case 3: $\tan \beta\ell = 0$

When $\tan \beta\ell = 0$, $\beta\ell = n\pi$ where n is an whole number (since we are considering positive lengths). Then, since $\beta = \frac{2\pi}{\lambda}$

$$\ell = \frac{n\lambda}{2} \ .$$

Now, when $\tan \beta\ell = 0$, from the fact that $Z_{\text{in}} = jZ_0 \tan \beta\ell$ for this lossless short-circuit line, we see that $Z_{\text{in}} = 0$. The input impedance being zero leads to the equivalent of a *series resonant L-C* circuit:

Case 4: $\tan \beta\ell \to \pm\infty$

When $\tan \beta\ell \to \pm\infty$, $\beta\ell = [(2n+1)\pi)]/2$ where n is an whole number (since we are considering positive lengths). Then, since $\beta = \frac{2\pi}{\lambda}$

$$\ell = \frac{(2n+1)\lambda}{4} \ .$$

Now, when $\tan \beta\ell \to \pm\infty$, from the fact that $Z_{\text{in}} = jZ_0 \tan \beta\ell$ for this lossless short-circuit line, we see that Z_{in} becomes infinite. The input impedance being infinite leads to the equivalent of a *parallel resonant L-C* circuit (you have seen this much earlier in your circuit theory career!):

Summary for a short-circuited lossless line: (1) When $0 < \ell < \frac{\lambda}{4}$ particular values of *inductive* reactance may be realized; (2) When $\frac{\lambda}{4} < \ell < \frac{\lambda}{2}$ particular values of *capacitive* reactance may be realized; (3) When $\ell = \frac{n\lambda}{2}$ a series resonant L-C equivalent circuit is obtained; and (4) When $\ell = \frac{(2n+1)\lambda}{4}$ a parallel resonant L-C equivalent circuit is obtained. Of course, if the resonant lengths are added to values of ℓ in cases (1) and (2), no change is made in the reactance from that of the shorter lines.

4.2.2 The Open-Circuit Line

Consider, as shown, an em open-circuited lossless line:

This time, the load impedance $Z_L \to \infty$. Therefore,

and the input impedance becomes

Again, Z_{in} looks purely reactive, and it is easy to be convinced that the lengths at which the open-circuit line looks capacitive/inductive is the same as the lengths for which the short-circuit line looks inductive/capacitive, respectively (CHECK IT OUT). Also, 0's and ∞'s become interchanged – i.e., the lengths ℓ for which $Z_{\text{in}} = 0$ on the open-circuit line are the same as those for which the short-circuit line has $Z_{\text{in}} \to \infty$ and vice versa.

Application: The half-wave dipole antenna

For the open-circuit lossless line, let $\beta\ell = \frac{\pi}{2}$ and since $\cot\frac{\pi}{2} = 0$, $Z_{\text{in}} = 0$ and we have the case of an equivalent series resonant L-C circuit. Also, $\ell = \frac{\lambda}{4}$.

Let's consider "opening out" the line as shown:

The result is a first attempt at modelling a "dipole antenna" and if the lengths are $\frac{\lambda}{4}$ we would expect it to be resonant (in actuality, this is the case if the wires are trimmed a little). Of course, if $l < \frac{\lambda}{4}$ the result looks capacitive and if $l > \frac{\lambda}{4}$ it looks inductive. Connecting the resonant dipole to a transmission line will allow the structure to radiate. The antenna can be modelled as shown. Note that R_r is called the *radiation resistance* of the antenna – it does NOT represent an ohmic element but is an equivalent "virtual" resistance giving rise to an average radiated power of $P_r = \frac{1}{2}|I|^2 R_r$ when the antenna carries a current I. For the $\lambda/2$ dipole structure it is possible to show that $R_r = 73\,\Omega$. This will be addressed in an elective course on "Antennas" in Term 7.

4.3 Graphical Methods for Transmisison Lines – The Smith Chart

The Equations Producing the Smith Chart:

The procedures in the earlier sections of this unit obviously involve the manipulation of complex numbers. To help reduce the tedium associated with this, P. H. Smith (1939) developed a graphical technique for analyzing and designing transmission-line circuits. While the resulting "Smith Chart" may be used for both lossy and lossless TL's we will confine our attention to the latter. Consider such a lossless line of characteristic impedance, Z_0, which is connected to a load of impedance Z_L.

The reflection coefficient is given as usual by

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{4.41}$$

Since the chart is to be useful for *any* lossless line and load impedance, it is important to *normalize* Z_L by Z_0. The result is represented by the lower case z_L as

$$z_L = \frac{Z_L}{Z_0} \; .$$

The reflection coefficient may then be written in terms of the normalized load impedance:

$$\Gamma = \frac{(Z_L/Z_0) - (Z_0/Z_0)}{(Z_L/Z_0) + (Z_0/Z_0)} = \frac{z_L - 1}{z_L + 1} \tag{4.42}$$

Now, this Γ may be used without direct reference to the characteristic impedance of the line as long as the normalized load impedance is used. Equation (4.42) may be solved for Z_L to give

$$z_L = \frac{1 + \Gamma}{1 - \Gamma} \tag{4.43}$$

In general, z_L will consist of a normalized load resistance, r_L, and a normalized load reactacnce, x_L:

$$z_L = r_L + jx_L \; . \tag{4.44}$$

Of course, Γ itself is in general complex and may be written as

$$\Gamma = \Gamma_r + j\Gamma_i \tag{4.45}$$

where Γ_r and Γ_i are the real and imaginary parts, respectively, of Γ. From equations (4.43) (4.44) and (4.45)

$$r_L + jx_L = \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i} \tag{4.46}$$

This allows us to find r_L and x_L in terms of Γ_r and Γ_i. To do this we simply multiply the numerator and denominator of (4.46) by the complex conjugate of the denominator to get (on separating the result into real and imaginary parts)

$$r_L = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} \tag{4.47}$$

and

$$x_L = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2} \tag{4.48}$$

It is easy to rearrange (4.47) and (4.48) to give, respectively,

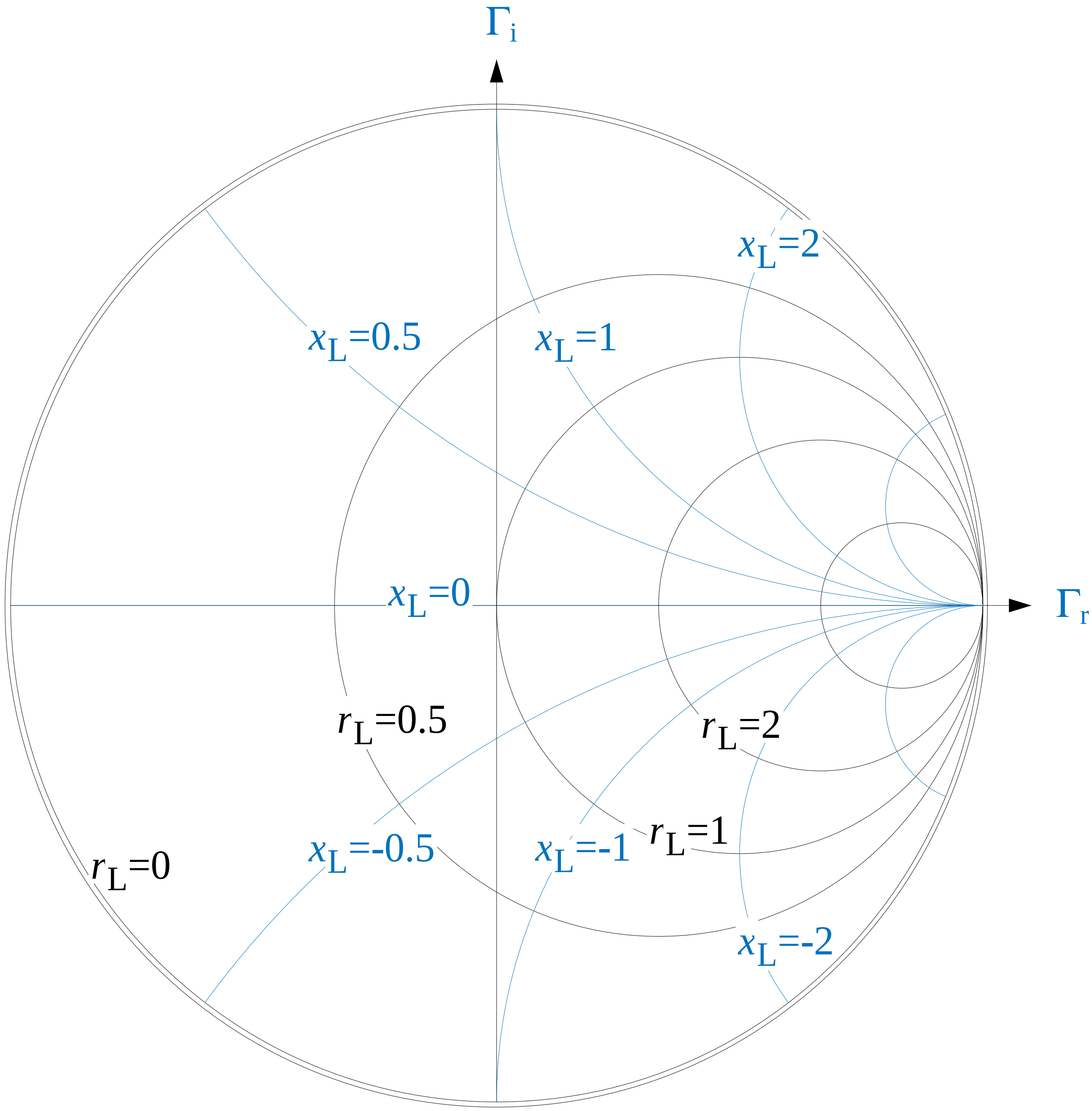
$$\left(\Gamma_r - \frac{r_L}{1+r_L}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r_L}\right)^2 \tag{4.49}$$

and

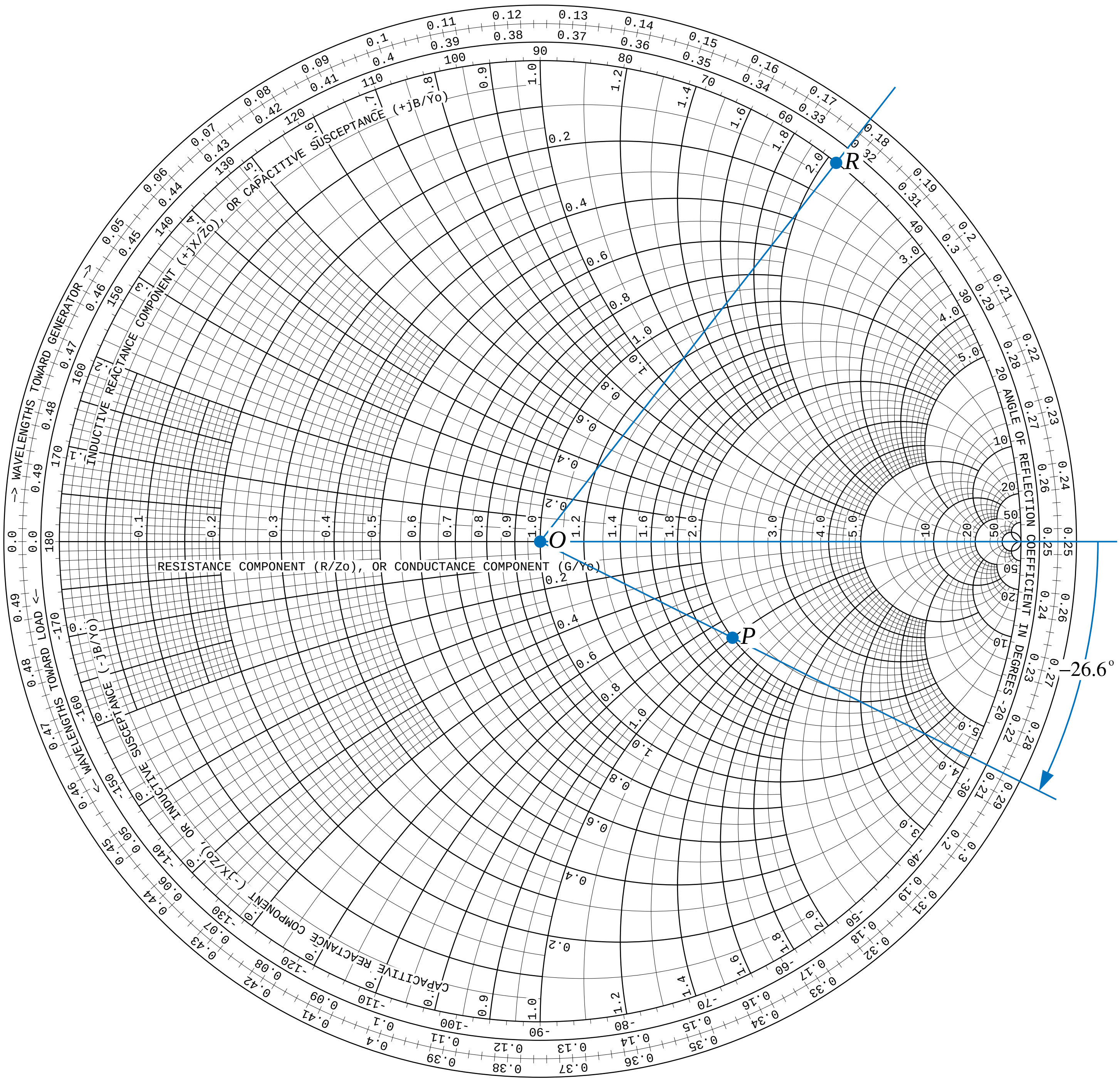
$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2 \tag{4.50}$$

Each of equations (4.49) and (4.50) are circles in the Γ_r-Γ_i plane Additionally, we note that it is easily verified that, if the load is passive, $|\Gamma| \leq 1$. Thus, the largest circle shown in the figure below is the unit circle. For the family of circles associated with x_L only a portion of each will fall with the bounds of the unit circle. Also, there are two families of x_L curves since that parameter can assume both positive and negative values. These curves appear on the Smith chart.

The points of intersection of the r_L and x_L curves obviously specify a particular reflection coefficient, Γ, for a given z_L. For example, in the second figure, the point P represents a *normalized* load impedance $z_L = 2 - j1$ with a corresponding voltage reflection coefficient of $\Gamma = 0.45e^{-j26.6^\circ}$ (remember, the largest value of $|\Gamma|$ is 1). Thus for a given Γ, z_L can be read off the chart and vice versa.

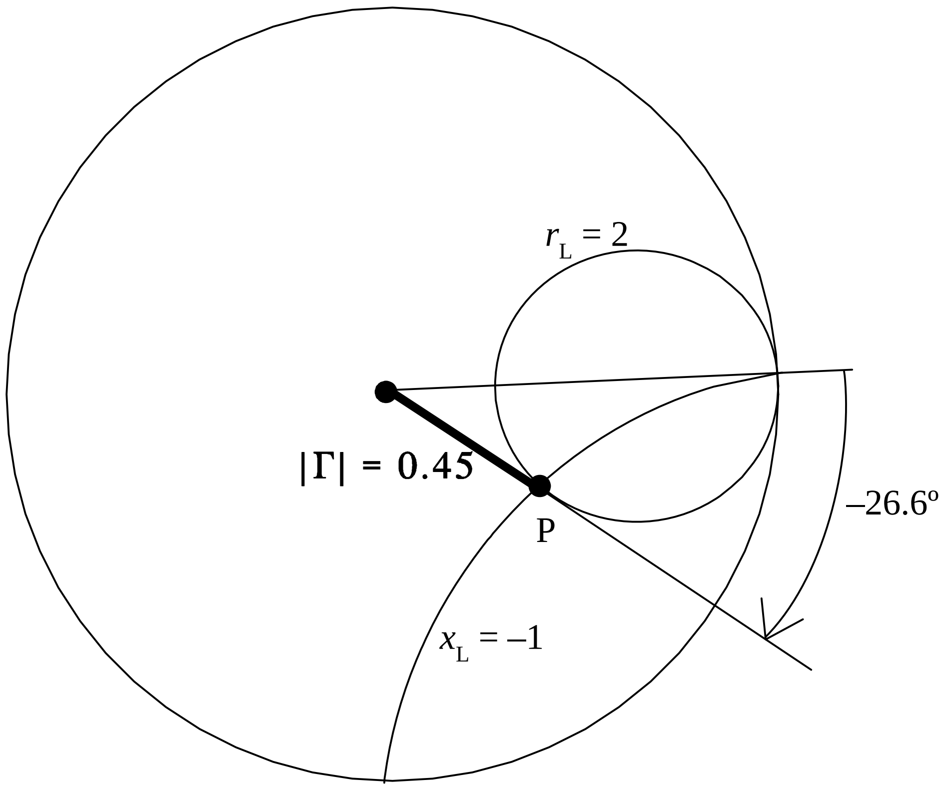


Γi
Γr
xL=2
xL=0.5
xL=1
xL=0
rL=0.5
rL=2
rL=1
rL=0
xL=-0.5
xL=-1
xL=-2



R
O
P
−26.6°
—> WAVELENGTHS TOWARD GENERATOR —>
<— WAVELENGTHS TOWARD LOAD <—
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES

21b



At P, $z_L = 2 - j1$. For a 50Ω line, the unnormalized load impedance would be Z_L = 100 – j50 Ω.

The Input Impedance:

From equations (4.31) and (4.32), it is readily verified that the input impedance is given by (SHOW THIS)

$$Z_{\text{in}} = Z_0 \left[\frac{1 + \Gamma e^{-j2\beta\ell}}{1 - \Gamma e^{-j2\beta\ell}} \right] \tag{4.51}$$

and normalizing by Z_0 we define

$$z_{\text{in}} = \frac{Z_{\text{in}}}{Z_0} = \left[\frac{1 + \Gamma e^{-j2\beta\ell}}{1 - \Gamma e^{-j2\beta\ell}} \right] \tag{4.52}$$

Since $\Gamma = |\Gamma| e^{j\theta_\Gamma}$, we define a *phase shifted* voltage reflection coefficient (as seen at $z = -\ell$) as

$$\Gamma_\ell = \Gamma e^{-j2\beta\ell} = |\Gamma| e^{(j\theta_\Gamma - j2\beta\ell)}$$

We note that while Γ_ℓ is phase shifted from Γ, the magnitudes are identical – remember, the line is lossless. With this new definition in place, we may now write

$$z_{\text{in}} = \frac{1 + \Gamma_\ell}{1 - \Gamma_\ell} \tag{4.53}$$

which suggests from (4.43) that if Γ is transformed into Γ_ℓ, then z_ℓ is transformed into z_{in}. On the Smith chart, to transform from Γ to Γ_ℓ, $|\Gamma|$ is maintained constant and a rotation of $2\beta\ell$ *clockwise* takes place. Noting that a complete rotation around the chart is equal to a phase change of 2π, the transmission line length corresponding to this is $\ell = \frac{\lambda}{2}$ (by setting $2\beta\ell = 2\frac{2\pi}{\lambda}\ell = 2\pi$). The outermost scale on the Smith chart is labelled in "wavelengths toward generator" (WTG) to denote a movement on the transmission line toward the generator (in units of λ, with once around, as noted, being $\lambda/2$).

If it is desired to determine Γ_ℓ, and subsequently z_{in}, from some position on the TL to a point *closer* to the load, then a similar procedure as above is used, but the rotation is counterclockwise on the "wavelengths toward load" (WTL) circle.

Finally, we note that because the SWR is given by

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|}\,, \tag{4.54}$$

a constant value of $|\Gamma|$ (whose locus is a circle) corresponds to a particular SWR. Thus, converting between normalized load impedance, z_L, and normalized input impedance,

z_{in}, or vice versa, we simply stay on the appropriate SWR circle and execute the appropriate rotation in WTG or WTL. Of course, to unnormalize the impedances we simply multiply by Z_0.

We have now laid the basis for the procedures involved in matching the characteristic impedance, Z_0, of a transmission line to a load, Z_L, so that maximum power transmission can occur. Read Section 13.5 of the text.