Eigenvalues and Eigenvectors for 2×2 Matrices

Let the general 2×2 matrix be $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad \det A = ad - bc$.

Define $D = (a-d)^2 + 4bc$

The eigenvalues of the matrix A are $\lambda = \dfrac{(a+d) \pm \sqrt{D}}{2}$ (for any choices of a, b, c, d)

Proof:

The characteristic equation is $\det(\lambda I - A) = 0 \quad \Rightarrow \quad \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = 0$

$\Rightarrow \quad (\lambda - a)(\lambda - d) - bc = 0 \quad \Rightarrow \quad \lambda^2 - (a+d)\lambda + (ad - bc) = 0$

$\Rightarrow \quad \lambda = \dfrac{+(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

But $(a+d)^2 - 4(ad-bc) = a^2 + 2ad + d^2 - 4ad + 4bc$

$= a^2 - 2ad + d^2 + 4bc = (a-d)^2 + 4bc = D$

If at least one of b or c is zero, (so that the matrix A is triangular), then the eigenvalues are just the diagonal elements, $\lambda = a$ or d (from $(\lambda - a)(\lambda - d) - bc = 0$ above).

The eigenvectors for the two eigenvalues are found by solving the underdetermined linear system

$A\vec{\mathbf{x}} = \lambda \vec{\mathbf{x}} \quad \Rightarrow \quad (\lambda I - A)\vec{\mathbf{x}} = \vec{\mathbf{0}}$

Let $\vec{\mathbf{x}} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, then $\begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Various cases arise.

<u>If $b = c = 0$</u> (so that the matrix A is diagonal), then:

For $\lambda = a$, $\begin{bmatrix} 0 & 0 \\ 0 & a-d \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad (a-d)\beta = 0$ and an eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For $\lambda = d$, $\begin{bmatrix} d-a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad (d-a)\alpha = 0$ and an eigenvector is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

<u>If $b = 0$ but $c \neq 0$</u> (so that the matrix A is lower triangular but not diagonal), then:

For $\lambda = a$, $\begin{bmatrix} 0 & 0 \\ -c & a-d \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow (a-d)\beta = c\alpha$ and an eigenvector is $\begin{bmatrix} a-d \\ c \end{bmatrix}$.

For $\lambda = d$, $\begin{bmatrix} d-a & 0 \\ -c & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c\alpha = 0$ and an eigenvector is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Note that if $b = 0$, $d = a$ and $c \neq 0$, then there is a single eigenvalue of multiplicity 2 with only one linearly independent eigenvector, so that the matrix A cannot be diagonalized.

<u>If $b \neq 0$ but $c = 0$</u> (so that the matrix A is upper triangular but not diagonal), then:

For $\lambda = a$, $\begin{bmatrix} 0 & -b \\ 0 & a-d \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow b\beta = 0$ and an eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For $\lambda = d$, $\begin{bmatrix} d-a & -b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow (d-a)\alpha = b\beta$ and an eigenvector is $\begin{bmatrix} b \\ d-a \end{bmatrix}$.

Note that if $b \neq 0$, $d = a$ and $c = 0$, then there is a single eigenvalue of multiplicity 2 with only one linearly independent eigenvector, so that the matrix A cannot be diagonalized.

<u>If $b \neq 0$ and $c \neq 0$</u> (so that the matrix A is not triangular), then:

For each eigenvalue λ, $\begin{bmatrix} \lambda-a & -b \\ -c & \lambda-d \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} (\lambda-a)\alpha - b\beta = 0 \\ -c\alpha + (\lambda-d)\beta = 0 \end{cases}$

An eigenvector is $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b \\ \lambda-a \end{bmatrix}$ or $\begin{bmatrix} \lambda-d \\ c \end{bmatrix}$ (which are multiples of each other).

The third case is actually a special case of the fourth and can be absorbed into it, so that
<u>If $b \neq 0$</u> (for any c), then:

For each eigenvalue λ, an eigenvector is $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b \\ \lambda-a \end{bmatrix}$.

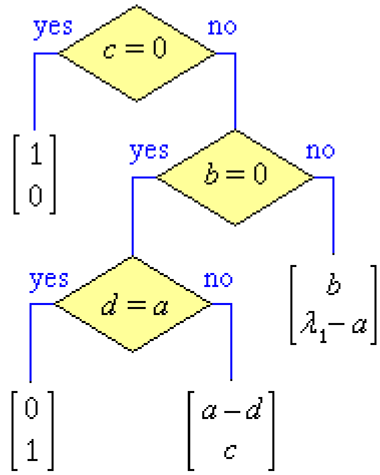
Similarly,
<u>If $c \neq 0$</u> (for any b), then:

For each eigenvalue λ, an eigenvector is $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \lambda-d \\ c \end{bmatrix}$.

The above can be rearranged for implementation in a spreadsheet
(`http://www.engr.mun.ca/~ggeorge/programs/MatrixMult.xls`).

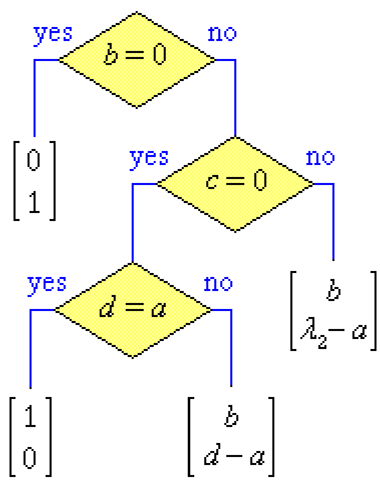
Flowchart for the eigenvector $\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$ for eigenvalue $\lambda_1 = \dfrac{(a+d)+\sqrt{D}}{2}$:

[Note: if and only if A is triangular, then $\lambda_1 = a$]



Flowchart for the eigenvector $\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$ for eigenvalue $\lambda_2 = \dfrac{(a+d)-\sqrt{D}}{2}$:

[Note: if and only if A is triangular, then $\lambda_2 = d$]



Whenever P^{-1} exists, the matrix $P = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix}$ is such that $\Lambda = P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

(a diagonal matrix).
