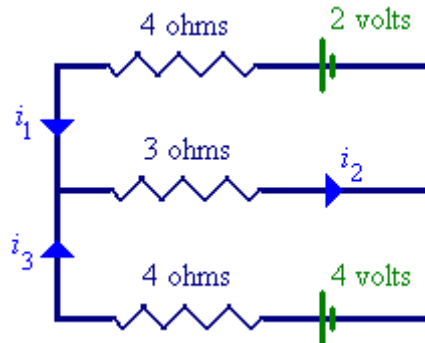


1. Systems of Linear Equations

Linear equations occur in many Engineering applications.

Example 1.01

Find the currents i_1 , i_2 , i_3 that flow through this circuit.



Kirchoff's laws lead to the linear system of equations

$$i_1 - i_2 + i_3 = 0 \quad (\text{KCL: sum of currents at any point is zero})$$

$$\left. \begin{aligned} 4i_1 + 3i_2 &= 2 \\ 3i_2 + 4i_3 &= 4 \end{aligned} \right\} \quad (\text{KVL: sum of voltage drops around any closed loop is zero})$$

Example 1.02

Find a combination of forces \mathbf{A} , \mathbf{B} , \mathbf{C} , on the particle such that the resultant force is the zero vector \mathbf{O} .

$$\bar{\mathbf{A}} = 14\hat{\mathbf{j}}, \quad \bar{\mathbf{B}} = 7\hat{\mathbf{i}} - 7\hat{\mathbf{j}}, \quad \bar{\mathbf{C}} = -10\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$$

We are seeking constants k_1 , k_2 , k_3 , such that $k_1\mathbf{A} + k_2\mathbf{B} + k_3\mathbf{C} = \mathbf{O}$.

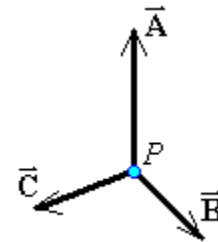
This leads to the linear system

$$7k_2 - 10k_3 = 0$$

and

$$14k_1 - 7k_2 - 4k_3 = 0$$

We shall return to these two examples later.



Determination of the intersections of planes in three-dimensional space (\mathbb{R}^3) leads to systems of linear equations in three variables.

Example 1.03

Show that the points $(-2, 3, 0)$ and $(0, -1, 2)$ both lie on all three of the planes

$$x + 2y + 3z = 4, \quad 4x + 3y + 2z = 1 \quad \text{and} \quad x + y + z = 1$$

and show that every member of the set of points $(t - 2, 3 - 2t, t)$ lies on all three planes, where t is any real number.

Setting $(x, y, z) = (-2, 3, 0)$, we find that

$$-2 + 2(3) + 3(0) = 4, \quad 4(-2) + 3(3) + 2(0) = 1 \quad \text{and} \quad -2 + 3 + 0 = 1$$

so that the point $(-2, 3, 0)$ lies on all three planes.

Setting $(x, y, z) = (0, -1, 2)$, we find that

$$0 + 2(-1) + 3(2) = 4, \quad 4(0) + 3(-1) + 2(2) = 1 \quad \text{and} \quad 0 - 1 + 2 = 1$$

so that the point $(0, -1, 2)$ also lies on all three planes.

$(x, y, z) = (-2, 3, 0)$ and $(x, y, z) = (0, -1, 2)$ are therefore two distinct **solutions** to the linear system

$$x + 2y + 3z = 4, \quad 4x + 3y + 2z = 1 \quad \text{and} \quad x + y + z = 1.$$

Setting $(x, y, z) = (t - 2, 3 - 2t, t)$, we find that

$$(t - 2) + 2(3 - 2t) + 3(t) = -2 + 6 + (1 - 4 + 3)t = 4,$$

$$4(t - 2) + 3(3 - 2t) + 2(t) = -8 + 9 + (4 - 6 + 2)t = 1$$

$$\text{and} \quad (t - 2) + (3 - 2t) + (t) = -2 + 3 + (1 - 2 + 1)t = 1$$

Therefore *every* point (x, y, z) of the form $(t - 2, 3 - 2t, t)$ is on all three planes, no matter what value of t is chosen.

Note that $(-2, 3, 0)$ corresponds to $t = 0$ and that $(0, -1, 2)$ corresponds to $t = 2$.

t is a **parameter** and the set of solutions $\{(x, y, z) = (t - 2, 3 - 2t, t)\}$ is the **general solution** to the system, expressed in **parametric form**.

This is a one-parameter family of solutions, corresponding to the one-dimensional line of intersection of the three planes.

In general, a linear system of p equations involving the same variables x_1, x_2, \dots, x_n is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{p1}x_1 + a_{p2}x_2 + a_{p3}x_3 + \dots + a_{pn}x_n &= b_p \end{aligned}$$

This is a linear system of p equations in n unknowns or, more briefly, a $p \times n$ system.

The number of equations is given first and the number of unknowns is indicated second. The constants a_{ij} are the **coefficients** and the constants b_i are the **right side constants**. This system can be abbreviated as

$$\sum_{j=1}^n a_{ij} x_j = b_i, \quad (i=1, 2, \dots, p)$$

or as the **augmented matrix**

$$[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{p1} & a_{p2} & a_{p3} & \cdots & a_{pn} & b_p \end{array} \right]$$

In general, a linear system has

- (i) infinitely many solutions; or
- (ii) a unique solution; or
- (iii) no solution at all.

Example 1.04

Solve the linear system $x + y = 0$ and $x + y = 1$.

It is impossible for the sum of two numbers to have two different values simultaneously. Therefore the linear system has no solution. The system is said to be **inconsistent**.

A linear system that has solutions is **consistent**.

Elementary Operations

Applying any of the following elementary operations to a linear system does not change the set of solutions:

- I. Interchange two equations.
- II. Multiply one equation by a non-zero number.
- III. Add a multiple of one equation to a different equation.

Each of these elementary operations has an inverse of the same type.

Operation I (swap equations p and q) is its own inverse.

The inverse of operation II (multiply equation p by $k (\neq 0)$) is (multiply equation p by $1/k$).

The inverse of operation III (add k times equation p to equation q) is (add $-k$ times equation p to equation q).

Thus any sequence of elementary operations can be undone to restore the original system. This leads to a systematic method for solving a linear system, **Gaussian elimination**.

Example 1.01 (again)

The linear system for the electric circuit is

$$i_1 - i_2 + i_3 = 0$$

$$4i_1 + 3i_2 = 2$$

$$3i_2 + 4i_3 = 4$$

for which the equivalent augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 2 \\ 0 & 3 & 4 & 4 \end{array} \right]$$

First, use operation III to clear all mention of i_1 from the latter two equations / rows.

Example 1.01 (continued)

To equation (2) add $(-4) \times$ equation (1):

$$\begin{array}{l} i_1 - i_2 + i_3 = 0 \\ 0i_1 + 7i_2 - 4i_3 = 2 \\ 3i_2 + 4i_3 = 4 \end{array} \quad \text{or} \quad \xrightarrow{R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 2 \\ 0 & 3 & 4 & 4 \end{array} \right]$$

Use operation II to make the first non-zero entry of equation (2) a '1'.
Multiply equation (2) by $1/7$:

$$\begin{array}{l} i_1 - i_2 + i_3 = 0 \\ i_2 - \frac{4}{7}i_3 = \frac{2}{7} \\ 3i_2 + 4i_3 = 4 \end{array} \quad \text{or} \quad \xrightarrow{R_2 \times \frac{1}{7}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & \frac{2}{7} \\ 0 & 3 & 4 & 4 \end{array} \right]$$

Use operation III to clear all mention of i_2 from the last equation / row.

To equation (3) add $(-3) \times$ equation (2):

$$\begin{array}{l} i_1 - i_2 + i_3 = 0 \\ i_2 - \frac{4}{7}i_3 = \frac{2}{7} \\ 0i_2 + \frac{40}{7}i_3 = \frac{22}{7} \end{array} \quad \text{or} \quad \xrightarrow{R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & \frac{2}{7} \\ 0 & 0 & \frac{40}{7} & \frac{22}{7} \end{array} \right]$$

Use operation II to make the first non-zero entry of equation (3) a '1'.
Multiply equation (3) by $7/40$:

$$\begin{array}{l} i_1 - i_2 + i_3 = 0 \\ i_2 - \frac{4}{7}i_3 = \frac{2}{7} \\ 1i_3 = \frac{11}{20} \end{array} \quad \text{or} \quad \xrightarrow{R_3 \times \frac{7}{40}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & \frac{2}{7} \\ 0 & 0 & 1 & \frac{11}{20} \end{array} \right]$$

The third equation now yields the value of i_3 immediately: $i_3 = 0.55$ A.

A process of **back-substitution** up the other equations yields the other values:

$$i_2 - \frac{4}{7}i_3 = \frac{2}{7} \quad \Rightarrow \quad i_2 = \frac{2}{7} + \frac{4}{7} \times \frac{11}{20} = \frac{10+11}{35} = \frac{21}{35} = \frac{3}{5}$$

and

$$i_1 - i_2 + i_3 = 0 \quad \Rightarrow \quad i_1 = \frac{3}{5} - \frac{11}{20} = \frac{12-11}{20} = \frac{1}{20}$$

Therefore the unique solution is $(i_1, i_2, i_3) = \left(\frac{1}{20}, \frac{3}{5}, \frac{11}{20}\right)$.

Example 1.01 (continued)

One can check that this solution does satisfy all three equations in the linear system.

$$i_1 - i_2 + i_3 = \frac{1-12+11}{20} = 0$$

$$4i_1 + 3i_2 = \frac{1+9}{5} = \frac{10}{5} = 2$$

$$3i_2 + 4i_3 = \frac{9+11}{5} = \frac{20}{5} = 4$$

A **row-echelon matrix** has the following features:

- 1) All zero rows are at the bottom; and
- 2) The first non-zero entry in each row is a '1' (the **leading one**); and
- 3) Each leading one is to the right of all leading ones in the rows above it.

Example 1.05: Matrices in row-echelon form:

$$B = \begin{bmatrix} 1 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(where the asterisk * denotes any value whatsoever).

If, in addition,

- 4) Each leading one is the only non-zero entry in its column,
- then the matrix is a **reduced row-echelon matrix**.

Example 1.06: Matrices in reduced row-echelon form:

$$E = \begin{bmatrix} 1 & * & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(where the asterisk * denotes any value whatsoever).

Example 1.01 (continued)

Continuing the reduction of the row echelon matrix to its fully reduced form,

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & \frac{2}{7} \\ 0 & 0 & 1 & \frac{11}{20} \end{array} \right] \xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{7} & \frac{2}{7} \\ 0 & 1 & -\frac{4}{7} & \frac{2}{7} \\ 0 & 0 & 1 & \frac{11}{20} \end{array} \right]$$

$$\begin{array}{l} R_1 - \frac{3}{7}R_3 \\ R_2 + \frac{4}{7}R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{40-33}{140} \\ 0 & 1 & 0 & \frac{10+11}{35} \\ 0 & 0 & 1 & \frac{11}{20} \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{20} \\ 0 & 1 & 0 & \frac{3}{5} \\ 0 & 0 & 1 & \frac{11}{20} \end{array} \right]$$

from which the unique solution is obvious.

The **rank** of a matrix is the number of leading ones in its row-echelon form.

In Example 1.01,

$$[A|b] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 2 \\ 0 & 3 & 4 & 4 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} \boxed{1} & -1 & 1 & 0 \\ 0 & \boxed{1} & -\frac{4}{7} & \frac{2}{7} \\ 0 & 0 & \boxed{1} & \frac{11}{20} \end{array} \right]$$

$$\Rightarrow \text{rank } A = \text{rank } [A|b] = 3 = n.$$

In Example 1.05,

$$B = \begin{bmatrix} 1 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \text{rank } B = 3 \text{ but } n = 5; \text{ rank } C = \text{rank } D = 3 = n.$$

In Example 1.06,

$$E = \begin{bmatrix} 1 & * & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \text{rank } E = 3 = \text{rank } B; \text{ rank } F = 3 = \text{rank } C \text{ and rank } G = 3 = \text{rank } D.$$

Example 1.02 (again)

The linear system is

$$7k_2 - 10k_3 = 0$$

and

$$14k_1 - 7k_2 - 4k_3 = 0$$

Note that this is a linear system of only two equations in three unknowns; it is **underdetermined**.

An underdetermined system cannot have a unique solution.

The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 7 & -10 & 0 \\ 14 & -7 & -4 & 0 \end{array} \right]$$

Clearing each row and column to reach reduced row-echelon form,

$$\left[\begin{array}{ccc|c} 0 & 7 & -10 & 0 \\ 14 & -7 & -4 & 0 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 14 & -7 & -4 & 0 \\ 0 & 7 & -10 & 0 \end{array} \right]$$

$$\xrightarrow{R_1 \times \frac{1}{14}} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{2}{7} & 0 \\ 0 & 7 & -10 & 0 \end{array} \right] \xrightarrow{R_2 \times \frac{1}{7}} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{2}{7} & 0 \\ 0 & 1 & -\frac{10}{7} & 0 \end{array} \right]$$

$$\xrightarrow{R_1 + \frac{1}{2}R_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -\frac{10}{7} & 0 \end{array} \right]$$

An equivalent linear system is therefore

$$k_1 = k_3, \quad k_2 = \frac{10}{7}k_3 \quad \text{and} \quad k_3 \in \mathbb{R}$$

which is a one-parameter family of solutions.

The solution with the smallest positive integer coefficients is

$$(k_1, k_2, k_3) = (7, 10, 7).$$

rank $A = \text{rank } [A | b] = 2$ but $n = 3$.

This consistent system has a $(n - r) = (3 - 2) = 1$ parameter family of solutions.

Example 1.03 (again)

The augmented matrix for the intersection of the three planes

$$x + 2y + 3z = 4, \quad 4x + 3y + 2z = 1 \quad \text{and} \quad x + y + z = 1$$

is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right]$$

Reducing this system:

$$\begin{array}{l} R_2 - 4R_1 \\ R_3 - R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -5 & -10 & -15 \\ 0 & -1 & -2 & -3 \end{array} \right] \xrightarrow{R_2 \times -\frac{1}{5}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \end{array} \right]$$

$$\begin{array}{l} R_1 - 2R_2 \\ R_3 + R_2 \end{array} \rightarrow \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The leading variables are x and y . z is a free parameter (call it t).

Also $\text{rank } A = \text{rank } [A | b] = 2$ but $n = 3$.

We have another one parameter family of solutions.

The reduced row echelon form can be read as

$$x - t = -2, \quad y + 2t = 3 \quad \text{and} \quad z = t, \quad \text{where } t \text{ is any real number.}$$

Another way to write the solution set is $(x, y, z) = (t - 2, 3 - 2t, t)$, as before.

Note that an inconsistent system will have the final leading one of its row echelon form in the column for the right side constants.

Example 1.04 (again)

Solve the linear system $x + y = 0$ and $x + y = 1$.

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 1 \end{array} \right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} \end{array} \right]$$

The second row $\Rightarrow 0x + 0y = 1$, which is impossible.
The linear system is therefore inconsistent.

$$\text{rank } A = 1 \text{ but } \text{rank } [A | b] = 2$$

If the right side constants are all zero, then the system is **homogeneous**.
Otherwise it is inhomogeneous.

Example 1.02 is a homogeneous system: $[A|b] = \left[\begin{array}{ccc|c} 0 & 7 & -10 & 0 \\ 14 & -7 & -4 & 0 \end{array} \right]$.

Examples 1.01, 1.03 and 1.04 are inhomogeneous:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 2 \\ 0 & 3 & 4 & 4 \end{array} \right], \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right], \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 1 \end{array} \right].$$

All homogeneous systems are consistent (they are guaranteed at least one solution).
The zero vector is always a solution (the **trivial solution**) to a homogeneous system:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n &= 0 \\ &\vdots \\ a_{p1}x_1 + a_{p2}x_2 + a_{p3}x_3 + \dots + a_{pn}x_n &= 0 \end{aligned}$$

clearly has the solution $x_1 = x_2 = x_3 = \dots = x_n = 0$

It may also have other (non-trivial) solutions.

Example 1.02 is guaranteed to have non-trivial solutions. Its reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{10}{7} & 0 \end{array} \right]$$

k_1 and k_2 are leading variables (their columns contain leading ones).

k_3 is a non-leading variable and is assigned as a parameter, in terms of which k_1 and k_2 are expressed:

$$k_1 = t, \quad k_2 = \frac{10}{7}t \quad \text{and} \quad k_3 = t \in \mathbb{R}$$

Setting t equal to any non-zero value generates a non-trivial solution.

This homogeneous system is underdetermined (fewer equations than variables) and is therefore guaranteed to have at least one non-leading variable, leading to non-trivial solutions.

In general, *any underdetermined homogeneous system is guaranteed to have non-trivial solutions.*

Other homogeneous linear systems may have non-trivial solutions or may have only one unique solution (the trivial solution).

Example 1.07

In Cartesian coordinates (x, y) the general equation of a conic section is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

where a, b, c, d, e, f are constants and a, b, c are not all zero.

Show that there is at least one conic section that passes through any five points that are not all on the same line.

Let the coordinates of the five points be (p_i, q_i) for $i = 1, 2, 3, 4, 5$.

The conic section passes through the point (p_i, q_i) if and only if

$$ap_i^2 + bp_iq_i + cq_i^2 + dp_i + eq_i + f = 0$$

This generates a linear system of five equations in the six unknowns (a, b, c, d, e, f) .

The system is homogeneous and underdetermined, so that non-trivial solutions exist.

If $a = b = c = 0$, then all five points lie on the same line $dx + ey + f = 0$, (contrary to the requirement in the question). Therefore at least one of a, b, c is not zero.

Therefore there is at least one conic section that passes through any five points that are not all on the same line.

Summary of Concepts in Unit 1:

A $p \times n$ **system** (linear system of p equations involving the same n variables x_1, x_2, \dots, x_n) is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{p1}x_1 + a_{p2}x_2 + a_{p3}x_3 + \dots + a_{pn}x_n &= b_p \end{aligned}$$

and can be abbreviated as

$$\sum_{j=1}^n a_{ij} x_j = b_i, \quad (i=1, 2, \dots, p)$$

or as the **augmented matrix**

$$[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{p1} & a_{p2} & a_{p3} & \cdots & a_{pn} & b_p \end{array} \right]$$

The constants a_{ij} are the **coefficients** and the constants b_i are the **right side constants**.

In general, a linear system has

- (i) infinitely many solutions (consistent); or
- (ii) a unique solution (consistent); or
- (iii) no solution at all (inconsistent).

Gaussian elimination transforms a system into row-echelon form or reduced row-echelon form, using the **elementary operations**

- I. Interchange two equations;
- II. Multiply one equation by a non-zero number;
- III. Add a multiple of one equation to a different equation.

Starting with the first row and working downwards,

If the row is all zero, swap rows (I) to move it to the last row.

Consider only columns to the right of any leading ones in rows above the current row.

If the first entry in this row is zero, then use (I) to swap this row with a row below that has a non-zero entry.

Use (II) to make the first entry a leading one.

Use (III) to transform all other entries in that column to zero.

Move to the next row down, until the last row (or until all remaining rows are all zero).

In a **row-echelon** form,

- 1) All zero rows are at the bottom; and
- 2) The first non-zero entry in each row is a '1' (the **leading one**); and
- 3) Each leading one is to the right of all leading ones in the rows above it.

If, in addition,

- 4) Each leading one is the only non-zero entry in its column,

then the matrix is a **reduced row-echelon matrix**.

The **rank** of a matrix is the number of leading ones in its row echelon form.

$\text{rank } A < \text{rank } [A \mid b] \Rightarrow$ the system is inconsistent.

$\text{rank } A = \text{rank } [A \mid b] = n \Rightarrow$ the system is consistent and has a unique solution

$\text{rank } A = \text{rank } [A \mid b] = r < n \Rightarrow$ the system is consistent and has infinitely many solutions, (an $(n-r)$ parameter family of solutions)

$b = \mathbf{0} \Rightarrow$ the system is **homogeneous** and has at least the trivial solution $X = \mathbf{0}$.

An underdetermined linear system cannot have a unique solution.

An underdetermined homogeneous linear system has infinitely many solutions.

Additional ExamplesExample 1.08

Find the equation of the parabola that passes through the points (4, 0), (2, -2) and the origin.

The equation of the parabola is $y = ax^2 + bx + c$.

The equation is satisfied by all three points.

Point (4, 0): $16a + 4b + c = 0$

Point (2, -2): $4a + 2b + c = -2$

Point (0, 0): $0 + 0 + c = 0$

We could note that equation (3) immediately yields $c = 0$ and reduces the system to

$$16a + 4b = 0$$

$$4a + 2b = -2$$

Proceeding in a systematic way:

$$\left[\begin{array}{ccc|c} 16 & 4 & 1 & 0 \\ 4 & 2 & 1 & -2 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{R_1 \times \frac{1}{16}} \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & \frac{1}{16} & 0 \\ 4 & 2 & 1 & -2 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\xrightarrow{R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & \frac{1}{16} & 0 \\ 0 & 1 & \frac{3}{4} & -2 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{R_1 - \frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{8} & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} & -2 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\xrightarrow{\begin{array}{l} R_1 + \frac{1}{8}R_3 \\ R_2 - \frac{3}{4}R_3 \end{array}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Therefore the parabola that passes through all three points has the equation

$$y = \frac{1}{2}x^2 - 2x$$

Example 1.09

Textbook page 25 question 3(b)

Find conditions on the constant a such that the linear system

$$x + y + 3z = a$$

$$ax + y + 5z = 4$$

$$x + ay + 4z = a$$

has zero, one or infinitely many solutions.

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ a & 1 & 5 & 4 \\ 1 & a & 4 & a \end{array} \right] \xrightarrow[\begin{array}{l} R_2 - aR_1 \\ R_3 - R_1 \end{array}]{R_2 - aR_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & 1-a & 5-3a & 4-a^2 \\ 0 & a-1 & 1 & 0 \end{array} \right]$$

We could add row 2 to row 3 at this point, to approach row-echelon form more quickly for all a . At this point there are two cases to consider. Case $a = 1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 \end{array} \right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{array} \right]$$

and the last equation becomes $0x + 0y + 0z = 3$, which is impossible.

The system is inconsistent.

Or, completing the reduction to row-echelon form,

$$\xrightarrow{R_3 \div 3} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 3 & a \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array} \right]$$

$\Rightarrow \text{rank } A = 2$ but $\text{rank } [A | b] = 3 \Rightarrow$ the system is inconsistent.

Case $a \neq 1$:

$$\xrightarrow{R_2 \div (1-a)} \left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & 1 & \frac{5-3a}{1-a} & \frac{4-a^2}{1-a} \\ 0 & a-1 & 1 & 0 \end{array} \right]$$

$$\xrightarrow[\begin{array}{l} R_1 - R_2 \\ R_3 + (1-a)R_2 \end{array}]{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 - \frac{5-3a}{1-a} & a - \frac{4-a^2}{1-a} \\ 0 & 1 & \frac{5-3a}{1-a} & \frac{4-a^2}{1-a} \\ 0 & 0 & 6-3a & 4-a^2 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{1-a} & \frac{a-4}{1-a} \\ 0 & 1 & \frac{5-3a}{1-a} & \frac{4-a^2}{1-a} \\ 0 & 0 & 6-3a & 4-a^2 \end{array} \right]$$

Example 1.09 (continued)

If $a = 2$ then the system becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{1-2} & \frac{2-4}{1-2} \\ 0 & 1 & \frac{5-6}{1-2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

which is in row-echelon form, with $\text{rank } A = \text{rank } [A | b] = 2$ but $n = 3$, leading to a one-parameter family of solutions.

Otherwise,

$$\xrightarrow{R_3 \div (6-3a)} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{1-a} & \frac{a-4}{1-a} \\ 0 & 1 & \frac{5-3a}{1-a} & \frac{4-a^2}{1-a} \\ 0 & 0 & 1 & \frac{a+2}{3} \end{array} \right]$$

which is in row-echelon form, with $\text{rank } A = \text{rank } [A | b] = n = 3$, leading to a unique solution.

Therefore the system has:

no solution if and only if $a = 1$;
 infinitely many solutions if and only if $a = 2$; and
 a unique solution otherwise.

Additional note:

The one-parameter family of solutions when $a = 2$ is $(x, y, z) = (2 - 2t, -t, t)$, $t \in \mathbb{R}$.

The unique solution (when a is neither 1 nor 2) is

$$\begin{aligned} (x, y, z) &= \left(\frac{a-4}{1-a} + \frac{2}{1-a} \cdot \frac{a+2}{3}, \frac{4-a^2}{1-a} + \frac{3a-5}{1-a} \cdot \frac{a+2}{3}, \frac{a+2}{3} \right) \\ &= \left(\frac{5a-8}{3(1-a)}, \frac{a+2}{3(1-a)}, \frac{a+2}{3} \right) \end{aligned}$$