3.1 The Cofactor Expansion for Determinants

Every square matrix has a determinant. All matrices with zero determinant are singular. All matrices with non-zero determinant are invertible.

The determinant of a (1×1) matrix $A = \begin{bmatrix} a \end{bmatrix}$ is just $\det A = a$.

From section 2.3, the determinant of a (2×2) matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det A = ad - bc$.

The determinants of all higher-order matrices can be expressed in terms of lower-order determinants. Details are on pages 105 – 108 of the textbook.

<u>Example 3.1.1</u>

Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Expanding along the top row and noting alternating signs $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$,

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = +1\times\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\times\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\times\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45-48) - 2(36-42) + 3(32-35) = -3+12-9 = 0$$

Therefore this matrix A is singular (has no inverse).

Definitions:

Let the ((*n*–1)×(*n*–1)) submatrix A_{ij} be the matrix obtained by the deletion of row *i* and column *j* of the (*n*×*n*) matrix A . [det A_{ij} is sometimes known as the (*i*, *j*)-minor of A.]

The **(*i*,*j*)-cofactor** of an (*n*×*n*) matrix A is $\boxed{c_{ij}(A) = (-1)^{i+j}\det(A_{ij})}$

In Example 3.1.1,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \Rightarrow \quad c_{11} = (-1)^{1+1}\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = +(5\times 9 - 6\times 8) = 45-48 = -3,$$

$$c_{12} = (-1)^{1+2}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -(4\times 9 - 6\times 7) = -(36-42) = +6,$$

$$c_{13} = (-1)^{1+3}\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = +(4\times 8 - 5\times 7) = -(32-35) = +3,$$

$$\vdots$$

$$c_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -(1\times 6 - 3\times 4) = -(6-12) = +6, \text{ etc.}$$

For the ($n\times n$) matrix A, the cofactor expansion of $\det A$ along row i is then

$$\det A = a_{i1}c_{i1}(A) + a_{i2}c_{i2}(A) + \ldots + a_{in}c_{in}(A) = \sum_{j=1}^{n} a_{ij}c_{ij}(A)$$

Any row can be chosen for the expansion, as can any column j :

$$\det A = a_{1j}c_{1j}(A) + a_{2j}c_{2j}(A) + \ldots + a_{nj}c_{nj}(A) = \sum_{i=1}^{n} a_{ij}c_{ij}(A)$$

Choosing to expand down column 2 in Example 3.1.1,

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 2\times(-1)^{1+2}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5\times(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + 8\times(-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

$$= -2(36-42) + 5(9-21) - 8(6-12) = 12 - 60 + 48 = 0$$

Choose the row or column that has the most zero entries.
Where an entry is zero, the cofactor need not be evaluated.

Example 3.1.2

Find the determinant of $B = \begin{bmatrix} 1 & 9 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 7 & 18 & 1 & 5 \\ 1 & -4 & 0 & 2 \end{bmatrix}$.

Row 2 and column 3 share the greatest number of zeros.
Column 3 looks easier (its non-zero entry is a '1').

Expand the (4×4) determinant along column 3:

$$\det\begin{bmatrix} 1 & 9 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 7 & 18 & 1 & 5 \\ 1 & -4 & 0 & 2 \end{bmatrix} = 0 + 0 + 1\times(-1)^{3+3}\begin{vmatrix} 1 & 9 & 1 \\ 0 & 2 & 0 \\ 1 & -4 & 2 \end{vmatrix} + 0$$

Expand the new (3×3) determinant along row 2:

$$\begin{vmatrix} 1 & 9 & 1 \\ 0 & 2 & 0 \\ 1 & -4 & 2 \end{vmatrix} = 0 + 2\times(-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 0 = 2(2-1) = 4$$

If a (4×4) matrix has no zero entries, then the cofactor expansion requires the evaluation of four (3×3) determinants, each of which involves the evaluation of three (2×2) determinants, for a total of twelve (2×2) determinants.

As n increases, the number of (2×2) determinants that need to be evaluated in the cofactor expansion for an (n×n) matrix with no zero entries increases very rapidly:

n	# (2×2) determinants
2	1
3	3
4	12
5	60
6	360

The determinant of a **triangular square matrix** is just the product of the entries on the leading diagonal.

<u>Proof for all upper triangular (4×4) matrices</u>:

Find the determinant of $U = \begin{bmatrix} a & u & v & w \\ 0 & b & x & y \\ 0 & 0 & c & z \\ 0 & 0 & 0 & d \end{bmatrix}$.

Expand down column 1 repeatedly:

$$\det U = a(-1)^{1+1}\begin{vmatrix} b & x & y \\ 0 & c & z \\ 0 & 0 & d \end{vmatrix} + 0 + 0 + 0 = ab(-1)^{1+1}\begin{vmatrix} c & z \\ 0 & d \end{vmatrix} + 0 + 0$$
$$= ab(cd - 0) = abcd$$

For a square matrix A, if any of the following is true, then $\det A = 0$:
A row or column is all zeros. [This is obvious upon expanding along the zero row/col.]
Two rows are identical.
Two columns are identical.
One row is a multiple of another row.
One column is a multiple of another column.

Example 3.1.3

Evaluate $\det C = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 & 0 \\ 67 & e & 3 & 0 & 0 \\ 101 & 1111 & -47 & 2 & 0 \\ \pi & -23 & 601 & -\sqrt{2} & 1 \end{vmatrix}$

Matrix C is lower triangular $\Rightarrow$ $\det C = 1 \times 2 \times 3 \times 2 \times 1 = 12$

Example 3.1.4

Evaluate $\det D = \begin{vmatrix} 1 & -1 & 1 & 3 \\ -4 & 2 & -4 & 2 \\ 7 & 2 & 7 & 1 \\ 11 & 3 & 11 & 27 \end{vmatrix}$

Columns 1 and 3 of matrix D are identical $\Rightarrow$ $\det D = 0$

Effect of row operations on the determinant

I (interchange two rows) changes the sign of the determinant
II (multiply a row by $k \neq 0$) multiplies the determinant by k .
III (add a multiple of one row to another row) does not change the determinant

Also $\det A^{\mathrm{T}} = \det A$ for all square matrices A .
Therefore *column* operations have the same effect on the determinant as row operations.

Example 3.1.1 (again)

Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Use elementary row operations to carry matrix A towards row echelon form:

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \xrightarrow[R_3 - 7R_1]{R_2 - 4R_1} \det A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix}$$

Clearly $R_3 = 2R_2 \;\Rightarrow\; \det A = 0$.
One further row operation ($R_3 - 2R_2$) will carry row 3 to all zeros.

Example 3.1.5 (Textbook, page 114, exercises 3.1, question 1(o))

Compute $\det A = \begin{vmatrix} 1 & -1 & 5 & 5 \\ 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & 1 & 2 & -1 \end{vmatrix}$.

Use elementary row operations to carry the matrix to upper triangular form:

$$\begin{vmatrix} 1 & -1 & 5 & 5 \\ 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & 1 & 2 & -1 \end{vmatrix} \xrightarrow[\substack{R_3 + R_1 \\ R_4 - R_1}]{R_2 - 3R_1} \begin{vmatrix} 1 & -1 & 5 & 5 \\ 0 & 4 & -13 & -11 \\ 0 & -4 & 13 & 5 \\ 0 & 2 & -3 & -6 \end{vmatrix}$$

$$\xrightarrow[\substack{R_3 + R_2 \\ R_4 - \frac{1}{2}R_2}]{} \begin{vmatrix} 1 & -1 & 5 & 5 \\ 0 & 4 & -13 & -11 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & \frac{7}{2} & -\frac{1}{2} \end{vmatrix} \xrightarrow[R_3 \leftrightarrow R_4]{} -\begin{vmatrix} 1 & -1 & 5 & 5 \\ 0 & 4 & -13 & -11 \\ 0 & 0 & \frac{7}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & -6 \end{vmatrix}$$

$$\Rightarrow \quad \det A = -1 \times 4 \times \tfrac{7}{2} \times (-6) = +84$$

Example 3.1.6 (Textbook, page 114, exercises 3.1, question 6(a))

Compute $\det A = \begin{vmatrix} a & b & c \\ a+1 & b+1 & c+1 \\ a-1 & b-1 & c-1 \end{vmatrix}$.

Note that the sum of rows 2 and 3 is twice row 1, which suggests a zero determinant.

$$\begin{vmatrix} a & b & c \\ a+1 & b+1 & c+1 \\ a-1 & b-1 & c-1 \end{vmatrix} \xrightarrow{R_2+R_3} \begin{vmatrix} a & b & c \\ 2a & 2b & 2c \\ a-1 & b-1 & c-1 \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ a & b & c \\ a-1 & b-1 & c-1 \end{vmatrix} = 0$$

because rows 1 and 2 are now identical.

Example 3.1.7 (Textbook, page 114, exercises 3.1, question 7(a))

If $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -1$, compute $\begin{vmatrix} -x & -y & -z \\ 3p+a & 3q+b & 3r+c \\ 2p & 2q & 2r \end{vmatrix}$.

$$\begin{vmatrix} -x & -y & -z \\ 3p+a & 3q+b & 3r+c \\ 2p & 2q & 2r \end{vmatrix} = (-1)\times 2\begin{vmatrix} x & y & z \\ 3p+a & 3q+b & 3r+c \\ p & q & r \end{vmatrix}$$

$$\xrightarrow{R_2-3R_3} -2\begin{vmatrix} x & y & z \\ a & b & c \\ p & q & r \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} +2\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{} -2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -2\times -1 = 2$$

Example 3.1.8 (Textbook, page 115, exercises 3.1, question 16(c))

Find the value(s) of x for which matrix $A = \begin{bmatrix} 1 & x & x^2 & x^3 \\ x & x^2 & x^3 & 1 \\ x^2 & x^3 & 1 & x \\ x^3 & 1 & x & x^2 \end{bmatrix}$ is singular.

Use elementary row operations to carry the matrix to upper triangular form:

$$\det A = \begin{vmatrix} 1 & x & x^2 & x^3 \\ x & x^2 & x^3 & 1 \\ x^2 & x^3 & 1 & x \\ x^3 & 1 & x & x^2 \end{vmatrix} \xrightarrow[\substack{R_3 - x^2R_1 \\ R_4 - x^3R_1}]{R_2 - xR_1} \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 0 & 0 & 1-x^4 \\ 0 & 0 & 1-x^4 & x(1-x^4) \\ 0 & 1-x^4 & x(1-x^4) & x^2(1-x^4) \end{vmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_4} -\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1-x^4 & x(1-x^4) & x^2(1-x^4) \\ 0 & 0 & 1-x^4 & x(1-x^4) \\ 0 & 0 & 0 & 1-x^4 \end{vmatrix} = -(1-x^4)^3$$

Matrix A is singular if and only if $-(1-x^4)^3 = 0$.

The only real values of x for which this happens are $x = \pm 1$.

Block Matrices

If A and B are square matrices, then for all matrices X, Y of the appropriate dimensions, $\det \begin{bmatrix} A & X \\ O & B \end{bmatrix} = \det \begin{bmatrix} A & O \\ Y & B \end{bmatrix} = \det A \det B$.

<u>Example 3.1.9</u> (Textbook, page 115, exercises 3.1, question 10(a))

Compute $\det M = \begin{vmatrix} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$.

$M = \begin{bmatrix} A & X \\ O & B \end{bmatrix}$, where $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 5 \end{bmatrix}$, $X = \begin{bmatrix} 0 & -2 \\ 4 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

$\Rightarrow \quad \det M = \det A \det B$

$$\det A = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 5 \end{vmatrix} = 0 + (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} + 0 = +(5-2) = 3$$

$\det B = 3 - (-1) = 4$

$\Rightarrow \quad \det M = 3 \times 4 = 12$

3.2 Determinants and Inverse Matrices

For any set of square matrices of the same dimensions, the determinant of a product is the product of the determinants:

$$\det(AB) = (\det A)(\det B), \quad \det(ABC) = (\det A)(\det B)(\det C), \quad \text{etc.}$$

It then follows that

$$\boxed{\det\left(A^k\right) = \left(\det A\right)^k}$$

$$AA^{-1} = I \quad \Rightarrow \quad \det\left(AA^{-1}\right) = 1 \quad \Rightarrow \quad \det A \det\left(A^{-1}\right) = 1 \quad \Rightarrow$$

$$\boxed{\det\left(A^{-1}\right) = \frac{1}{\det A}}$$

The **adjugate** (or **adjoint**) of any square matrix A is the transpose of the matrix of cofactors of A:

$$\boxed{\operatorname{adj}(A) = \left[c_{ij}(A)\right]^{\mathrm{T}}} \qquad \text{[The } 2\times 2 \text{ case is } \operatorname{adj}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.\text{]}$$

Example 3.2.1

Compute adj (A), A adj (A) and det (A) for $A = \begin{bmatrix} 1 & 9 & 1 \\ 0 & 2 & 0 \\ 1 & -4 & 2 \end{bmatrix}$.

The matrix of cofactors is

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} +\begin{vmatrix} 2 & 0 \\ -4 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ 1 & -4 \end{vmatrix} \\ -\begin{vmatrix} 9 & 1 \\ -4 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 9 \\ 1 & -4 \end{vmatrix} \\ +\begin{vmatrix} 9 & 1 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 9 \\ 0 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 4 & 0 & -2 \\ -22 & 1 & 13 \\ -2 & 0 & 2 \end{bmatrix}$$

$$\operatorname{adj} A = C^{\mathrm{T}} = \begin{bmatrix} 4 & -22 & -2 \\ 0 & 1 & 0 \\ -2 & 13 & 2 \end{bmatrix}$$

Example 3.2.1 (continued)

$$A\,\text{adj}\,A = \begin{bmatrix} 1 & 9 & 1 \\ 0 & 2 & 0 \\ 1 & -4 & 2 \end{bmatrix} \begin{bmatrix} 4 & -22 & -2 \\ 0 & 1 & 0 \\ -2 & 13 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2I$$

Expand along the middle row to find $\det A$:
$\det A = 0 + 2 \times c_{22} + 0 = 2$

Note that $A\,\text{adj}\,(A) = (\det(A))\,I \quad \Rightarrow \quad A^{-1} = \dfrac{\text{adj}\,A}{\det A}$

The inverse of any non-singular matrix A is

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}$$

However, this is often a very inefficient way to compute the inverse of a matrix. Gaussian elimination of $[\,A \mid I\,]$ to $[\,I \mid A^{-1}\,]$ is usually much faster.

Example 3.2.2 (Textbook, page 127, exercises 3.2, question 2(e))

Use determinants to find which real value(s) of c make this matrix invertible:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & c \\ 2 & c & 1 \end{bmatrix}$$

$$A = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & c \\ 2 & c & 1 \end{vmatrix} = 0 + (-1)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - c\begin{vmatrix} 1 & 2 \\ 2 & c \end{vmatrix}$$

$= -(1+2) - c(c-4) = -(c^2 - 4c + 3) = -(c-1)(c-3)$

$\det A = 0 \quad \Rightarrow \quad c = 1$ or $c = 3$

Therefore the matrix is invertible for all real values of c except $c = 1$ or $c = 3$.

<u>Example 3.2.3</u> (Textbook, page 127, exercises 3.2, question 16)

Show that no 3×3 matrix A exists such that $A^2 + I = O$.
Find a 2×2 matrix A with this property.

$$A^2 + I = O \quad \Rightarrow \quad A^2 = -I \quad \Rightarrow \quad \det\left(A^2\right) = \det(-I)$$

$$\Rightarrow \quad (\det A)^2 = \det(-I)$$

But $-I$ is an (n×n) matrix whose only non-zero entries are the n entries of -1 down the main diagonal

$$\Rightarrow \quad \det(-I) = (-1)^n = \begin{cases} +1 & (\text{if } n \text{ is even}) \\ -1 & (\text{if } n \text{ is odd}) \end{cases}$$

For a (3×3) matrix we therefore have $(\det A)^2 = -1$, which has no real solution.

For a (2×2) matrix we have $(\det A)^2 = +1 \quad \Rightarrow \quad \det A = \pm 1$

Solving $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$\Rightarrow \quad d = -a$ and $bc = -(a^2 + 1)$.

The set of all (2×2) matrices satisfying $A^2 + I = O$ is

$$A = \begin{bmatrix} a & b \\ \dfrac{a^2+1}{-b} & -a \end{bmatrix} \quad \left(a \in \mathbb{R},\ b \in \mathbb{R},\quad b \neq 0\right)$$

One member of this set is $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Cramer's Rule

For a linear system of n equations in n unknowns, if the coefficient matrix A is invertible, then define the matrices A_k by replacing the i^{th} column of A by B and the unique solution of the linear system $AX = B$ is $X = [\, x_1 \;\; x_2 \;\; \ldots \;\; x_n]^T$, where

$$\boxed{x_i = \frac{\det A_k}{\det A}}$$

Example 3.2.4 (Textbook, page 127, exercises 3.2, question 8(c) modified)

Find the value of x when

$$\begin{aligned} 5x + y - z &= -7 \\ 2x - y - 2z &= 6 \\ 3x \quad + 2z &= -7 \end{aligned}$$

$$A = \begin{bmatrix} 5 & 1 & -1 \\ 2 & -1 & -2 \\ 3 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -7 \\ 6 \\ -7 \end{bmatrix} \quad \Rightarrow \quad A_1 = \begin{bmatrix} -7 & 1 & -1 \\ 6 & -1 & -2 \\ -7 & 0 & 2 \end{bmatrix}$$

Expanding down the middle column,

$$\det A_1 = \begin{vmatrix} -7 & 1 & -1 \\ 6 & -1 & -2 \\ -7 & 0 & 2 \end{vmatrix} = 1\times(-1)^{1+2}\begin{vmatrix} 6 & -2 \\ -7 & 2 \end{vmatrix} + (-1)\times(-1)^{2+2}\begin{vmatrix} -7 & -1 \\ -7 & 2 \end{vmatrix} + 0$$

$$= -(12-14) - (-14-7) = +2+21 = 23$$

$$\det A = \begin{vmatrix} 5 & 1 & -1 \\ 2 & -1 & -2 \\ 3 & 0 & 2 \end{vmatrix} = 1\times(-1)^{1+2}\begin{vmatrix} 2 & -2 \\ 3 & 2 \end{vmatrix} + (-1)\times(-1)^{2+2}\begin{vmatrix} 5 & -1 \\ 3 & 2 \end{vmatrix} + 0$$

$$= -(4+6) - (10+3) = -10-13 = -23$$

$$x = \frac{\det A_1}{\det A} = \frac{23}{-23} = -1$$

Therefore $x = -1$.

Cramer's rule is computationally a very inefficient method for solving linear systems.

Example 3.2.5 (Textbook, page 127, exercises 3.2, question 8(a))

Use Cramer's Rule to solve the system

$$\begin{aligned} 2x + y &= 1 \\ 3x + 7y &= -2 \end{aligned}$$

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \Rightarrow \quad A_1 = \begin{bmatrix} 1 & 1 \\ -2 & 7 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$$

$$\Rightarrow \quad \det A = 14 - 3 = 11, \quad \det A_1 = 7 + 2 = 9, \quad \det A_2 = -4 - 3 = -7$$

$$\Rightarrow \quad x = \frac{\det A_1}{\det A} = \underline{\underline{\frac{9}{11}}} \quad \text{and} \quad y = \frac{\det A_2}{\det A} = \underline{\underline{\frac{-7}{11}}}$$

Check 1:

$$A^{-1} = \frac{1}{14-3}\begin{bmatrix} 7 & -1 \\ -3 & 2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 7 & -1 \\ -3 & 2 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B = \frac{1}{11}\begin{bmatrix} 7 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 9 \\ -7 \end{bmatrix} \quad ✓$$

Check 2:

$$2\left(\tfrac{9}{11}\right) + \left(\tfrac{-7}{11}\right) = \tfrac{11}{11} = 1 \quad ✓$$

$$3\left(\tfrac{9}{11}\right) + 7\left(\tfrac{-7}{11}\right) = \tfrac{27-49}{11} = \tfrac{-22}{11} = -2 \quad ✓$$

3.3 – Eigenvalues and Eigenvectors

λ is an **eigenvalue** of an $(n \times n)$ matrix A if, for some column vector $X \neq O$,

$$AX = \lambda X$$

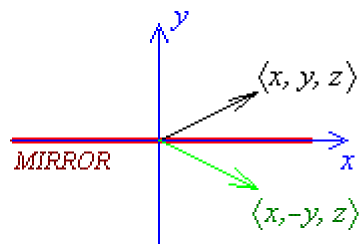
The non-trivial column vector X is an **eigenvector** of A for that eigenvalue, (as is any non-zero multiple of that column vector).

Example 3.3.1

$$AX = \begin{bmatrix} 1 & 1 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 30 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 5 \end{bmatrix} = 6X$$

$X = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ is therefore an eigenvector of A for eigenvalue $\lambda = 6$ (the 6-eigenvalue).

Example 3.3.2



A mirror is in the x-z plane in $\mathbb{R}^3$ space.
The vector from the origin to a general point (x, y, z) is $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.
The reflection of this vector in the mirror is the vector $(x\mathbf{i} - y\mathbf{j} + z\mathbf{k})$.
The operation of reflection may be represented by the matrix

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ because } RX = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -y \\ z \end{bmatrix}.$$

Any vector in the plane of the mirror, $(x\mathbf{i} + 0\mathbf{j} + z\mathbf{k})$, does not move upon reflection.

Therefore $X = [\, x \;\; 0 \;\; z \,]^T$ is an eigenvector of R for eigenvalue $\lambda = 1$ for any choices of x and z that are not both zero. Because $[\, x \;\; 0 \;\; z \,]^T = [\, 1\; 0\; 0 \,]^T x + [\, 0\; 0\; 1 \,]^T z$, the basic set of 1-eigenvectors is $\{ [\, 1\; 0\; 0 \,]^T, [\, 0\; 0\; 1 \,]^T \}$.

The general vector from the origin, proceeding out at right angles to the plane of the mirror along the y axis, is $(0\mathbf{i} + y\mathbf{j} + 0\mathbf{k})$.
Its reflection in the mirror is the vector $(0\mathbf{i} - y\mathbf{j} + 0\mathbf{k})$.

Therefore $X = [\, 0 \;\; y \;\; 0 \,]^T$ is an eigenvector of R for eigenvalue $\lambda = -1$ for any non-zero choice of y. The basic set of -1-eigenvectors is $\{ [\, 0 \;\; 1 \;\; 0 \,]^T \}$.

No other vectors are parallel to their own reflections in the mirror.

Characteristic Polynomial

If a non-zero column vector X is an eigenvector of ($n \times n$) matrix A for eigenvalue λ, then

$$AX = \lambda X \quad \Rightarrow \quad AX = \lambda IX \quad \Rightarrow \quad \lambda IX - AX = O \quad \Rightarrow \quad (\lambda I - A)X = O$$

But this square homogeneous linear system cannot have a non-trivial solution unless $(\lambda I - A)$ is singular $\Rightarrow$ $\det(\lambda I - A) = 0$.

The characteristic polynomial of any ($n \times n$) matrix A is $c_A(\lambda) = \det(\lambda I - A)$, which is a polynomial of degree n in λ. The eigenvalues of A are the n solutions of $c_A(\lambda) = \det(\lambda I - A) = 0$.

The λ-eigenvectors of A are the non-trivial solutions to the homogeneous linear system $(\lambda I - A)X = O$.

Example 3.3.1 (continued)

Find all eigenvalues and their eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ -5 & 7 \end{bmatrix}$.

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -5 & 7 \end{bmatrix}\right) = \begin{vmatrix} \lambda - 1 & -1 \\ 5 & \lambda - 7 \end{vmatrix} = (\lambda^2 - 8\lambda + 7) + 5$$

$$= \lambda^2 - 8\lambda + 12 = (\lambda - 2)(\lambda - 6)$$

$$\det(\lambda I - A) = 0 \quad \Rightarrow \quad \lambda = 2 \text{ or } \lambda = 6$$

$\lambda_1 = 2$:

Solving $(2I - A)X_1 = O$:

$$(2I - A)X = \begin{bmatrix} 2-1 & -1 \\ 5 & 2-7 \end{bmatrix} X = \begin{bmatrix} 1 & -1 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x - y = 0 \quad \Rightarrow \quad y = x$$

Therefore the 2-eigenvectors of A are any non-zero multiples of $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example 3.3.1 (continued)

$\lambda_2 = 6$ (which is the case considered earlier):
Solving $(6I - A)X_2 = O$:

$$(6I-A)X = \begin{bmatrix} 6-1 & -1 \\ 5 & 6-7 \end{bmatrix} X = \begin{bmatrix} 5 & -1 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 5x - y = 0 \qquad \Rightarrow \quad y = 5x$

Therefore the 6-eigenvectors of A are any non-zero multiples of $X_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Example 3.3.3

Find all eigenvalues and their eigenvectors of $A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$.

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}\right) = \begin{vmatrix} \lambda-2 & -1 \\ 0 & \lambda-4 \end{vmatrix} = (\lambda-2)(\lambda-4)$$

$\det(\lambda I - A) = 0 \qquad \Rightarrow \quad \lambda = 2$ or $\lambda = 4$

$\lambda_1 = 2$:
Solving $(2I - A)X = O$:

$$(2I-A)X = \begin{bmatrix} 2-2 & -1 \\ 0 & 2-4 \end{bmatrix} X = \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad y = 0, \quad (x \text{ arbitrary})$

Therefore the 2-eigenvectors of A are any non-zero multiples of $X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\lambda_2 = 4$:
Solving $(4I - A)X_2 = O$:

$$(4I-A)X = \begin{bmatrix} 4-2 & -1 \\ 0 & 4-4 \end{bmatrix} X = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 2x - y = 0 \qquad \Rightarrow \quad y = 2x$

Therefore the 4-eigenvectors of A are any non-zero multiples of $X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Eigenvalues and eigenvectors do not have to be real.

The rotation matrix in $\mathbb{R}^2$, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, has eigenvalues
$\lambda = \cos\theta \pm j\sin\theta = e^{\pm j\theta}$, $\left(\text{where } j = \sqrt{-1}\right)$.

An **upper triangular** matrix has all its non-zero entries on or above the main diagonal.
$\begin{bmatrix} 1 & 5 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is upper triangular.

A **lower triangular** matrix has all its non-zero entries on or below the main diagonal.
$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & -2 & -4 \end{bmatrix}$ is lower triangular.

The eigenvalues of a triangular matrix are just the entries on the main diagonal.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ 0 & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - a_{nn} \end{vmatrix} = (\lambda - a_{11})(\lambda - a_{22})\ldots(\lambda - a_{nn})$$

$$\det(\lambda I - A) = 0 \quad \Rightarrow \quad \lambda = a_{11}, a_{22}, \ldots, a_{nn}$$

The matrix in example 3.3.3 is upper triangular. We can say immediately that its eigenvalues are 2 and 4 (the entries on the main diagonal).

Example 3.3.4

Find all eigenvalues and their eigenvectors of $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$.

$$c_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -1 & -1 \\ 4 & \lambda+2 & 5 \\ -2 & -2 & \lambda-5 \end{vmatrix}$$

Expand this determinant along the top row:

$$c_A(\lambda) = (\lambda-3)\begin{vmatrix} \lambda+2 & 5 \\ -2 & \lambda-5 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 5 \\ -2 & \lambda-5 \end{vmatrix} + (-1)\begin{vmatrix} 4 & \lambda+2 \\ -2 & -2 \end{vmatrix}$$

$$= (\lambda-3)(\lambda^2-3\lambda-10+10) + (4\lambda-20+10) - (-8+2\lambda+4)$$

$$= (\lambda^2-6\lambda+9)\lambda + 2\lambda-6 = \lambda(\lambda-3)^2+2(\lambda-3)$$

$$= (\lambda-3)(\lambda^2-3\lambda+2) = (\lambda-1)(\lambda-2)(\lambda-3)$$

$c_A(\lambda)=0 \quad \Rightarrow \quad \boxed{\lambda=1 \text{ or } \lambda=2 \text{ or } \lambda=3}$

For $\lambda = 1$:
Solving $(1I-A)X=O$:

$$\begin{bmatrix} -2 & -1 & -1 \\ 4 & 3 & 5 \\ -2 & -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Carry the coefficient matrix to reduced row-echelon form:

$$\xrightarrow{R_1 \div (-2)} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 4 & 3 & 5 \\ -2 & -2 & -4 \end{bmatrix} \xrightarrow[R_3+2R_1]{R_2-4R_1} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$$\xrightarrow[R_3+R_2]{R_1-\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad x=z$ and $y=-3z$

The 1-eigenvector is therefore any non-zero multiple of $X_1 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$

Example 3.3.4 (continued)

For $\lambda = 2$:
Solving $(2I - A)X = O$:

$$\begin{bmatrix} -1 & -1 & -1 \\ 4 & 4 & 5 \\ -2 & -2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Carry the coefficient matrix to reduced row-echelon form:

$$\xrightarrow{R_1 \div (-1)} \begin{bmatrix} 1 & 1 & 1 \\ 4 & 4 & 5 \\ -2 & -2 & -3 \end{bmatrix} \xrightarrow[R_3 + 2R_1]{R_2 - 4R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow[R_3 + R_2]{R_1 - R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $x = -y$ and $z = 0$.

The 2-eigenvector is therefore any non-zero multiple of $X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

For $\lambda = 3$:
Solving $(3I - A)X = O$:

$$\begin{bmatrix} 0 & -1 & -1 \\ 4 & 5 & 5 \\ -2 & -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Carry the coefficient matrix to reduced row-echelon form:

$$\xrightarrow[\text{then } R_1 \div (-2)]{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 5 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_2 - 4R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow[R_3 + R_2]{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $x = 0$ and $y = -z$.

The 3-eigenvector is therefore any non-zero multiple of $X_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

The **multiplicity** of an eigenvalue is the number of times that distinct eigenvalue is repeated in the solution of the characteristic polynomial.

In Example 3.3.2, the three eigenvalues of R are –1, +1 and +1.
$\lambda = -1$ has multiplicity 1. $\lambda = +1$ has multiplicity 2.
In the other examples, all eigenvalues have multiplicity 1.

Each distinct eigenvalue has at least one basic eigenvector (and at most *m*, where *m* is the multiplicity of the eigenvalue).

If and only if an ($n \times n$) matrix has a total of n basic eigenvectors, then it can be diagonalized.

Diagonalization

A square matrix that is both upper and lower triangular is **diagonal**.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \operatorname{diag}(1, 3, -4) \text{ is diagonal.}$$

Diagonal matrices have nice properties.
If $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $E = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n)$ then
$D + E = \operatorname{diag}(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots, \lambda_n + \mu_n)$
and $DE = ED = \operatorname{diag}(\lambda_1\mu_1, \lambda_2\mu_2, \ldots, \lambda_n\mu_n)$

A square matrix A is diagonalizable iff an invertible matrix P exists such that $D = P^{-1}AP$ is a diagonal matrix.

Let $X_1, X_2, \ldots, X_n$ denote the columns of P, then we can write $P = [\, X_1 \ \ X_2 \ \ldots \ X_n \,]$
$D = P^{-1}AP \quad \Rightarrow \quad PD = PP^{-1}AP = IAP = AP$
$\Rightarrow \ [\, X_1 \ \ X_2 \ \ \cdots \ \ X_n \,] \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = AP$
$\Rightarrow \ [\, \lambda_1 X_1 \ \ \lambda_2 X_2 \ \ \cdots \ \ \lambda_n X_n \,] = A[\, X_1 \ \ X_2 \ \ \cdots \ \ X_n \,]$
$\Rightarrow \ \lambda_i X_i = AX_i \quad (i = 1, 2, \ldots, n)$
Therefore the main diagonal entries of D are the eigenvalues of A and each column of P is an eigenvector for the corresponding eigenvalue.

Example 3.3.5

Find the matrix P that diagonalizes $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$ and write down the diagonal matrix.

From Example 3.3.4, the eigenvalues and corresponding set of basic eigenvectors of A are:

$$\lambda_1 = 1, \quad X_1 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \quad \lambda_2 = 2, \quad X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \lambda_3 = 3, \quad X_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad P = \begin{bmatrix} 1 & 1 & 0 \\ -3 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

and

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

To verify this result, let us find P^{-1} by Gaussian elimination, then $P^{-1}AP$.

$$[P \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -3 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 + 3R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \div 2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + R_2]{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right]$$

$$\xrightarrow[\substack{\text{then} \\ R_3 \times (-2)}]{\substack{R_1 - R_3 \\ R_2 + R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & -2 \end{array}\right] = [I \mid P^{-1}]$$

Example 3.3.5 (continued)

and it is straightforward to verify that

$$P^{-1}P = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 1 & 1 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -3 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$AP = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ -3 & -2 & 3 \\ 1 & 0 & -3 \end{bmatrix}$$

$$\Rightarrow \quad P^{-1}AP = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 1 & 1 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ -3 & -2 & 3 \\ 1 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

Interchanging the order in which the eigenvalues are written in D also interchanges the corresponding columns in the diagonalizing matrix P.

$$D = P^{-1}AP \quad \Rightarrow \quad PDP^{-1} = PP^{-1}APP^{-1} = IAI = A$$

$$\Rightarrow \quad A^k = \left(PDP^{-1}\right)^k = \underbrace{\left(PDP^{-1}\right)\left(PDP^{-1}\right)\ldots\left(PDP^{-1}\right)}_{k \text{ factors}} = P\underbrace{DIDID\ldots ID}_{k \text{ factors}}P^{-1} = PD^kP^{-1}$$

It then follows that the eigenvalues of A^k are the k^{th} powers of the eigenvalues of A.
To find A^k quickly for high values of k, find the eigenvalues and eigenvectors, hence matrices D, P and P^{-1}, then $A^k = PD^kP^{-1}$.

Example 3.3.6

Find A^5, where $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$.

From Examples 3.3.4 and 3.3.5,

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 0 \\ -3 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 1 & 1 \\ -1 & -1 & -2 \end{bmatrix}$$

$$\Rightarrow \quad A^5 = PD^5P^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -3 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1^5 & 0 & 0 \\ 0 & 2^5 & 0 \\ 0 & 0 & 3^5 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ 2 & 1 & 1 \\ -1 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -3 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ 64 & 32 & 32 \\ -243 & -243 & -486 \end{bmatrix}$$

$$\Rightarrow \quad A^5 = \begin{bmatrix} 63 & 31 & 31 \\ -304 & -272 & -515 \\ 242 & 242 & 485 \end{bmatrix}$$

This can be verified by the tedious process of calculating

$$A^2 = AA = \begin{bmatrix} 7 & 3 & 3 \\ -14 & -10 & -19 \\ 8 & 8 & 17 \end{bmatrix}, \quad A^3 = A^2A = \begin{bmatrix} 15 & 7 & 7 \\ -40 & -32 & -59 \\ 26 & 26 & 53 \end{bmatrix} \text{ and finally}$$

$$A^5 = A^3A^2 = \begin{bmatrix} 63 & 31 & 31 \\ -304 & -272 & -515 \\ 242 & 242 & 485 \end{bmatrix}$$

Example 3.3.7 (Textbook, exercises 3.3, page 141, question 3)

Show that A has $\lambda = 0$ as an eigenvalue if and only if A is not invertible.

The characteristic equation for the eigenvalues is $\det(\lambda I - A) = 0$ which becomes $\left(\lambda - \lambda_1\right)\left(\lambda - \lambda_2\right)\ldots\left(\lambda - \lambda_n\right) = 0$.

If any one or more of the eigenvalues is zero, then $\det(0\,I - A) = -\det A = 0$
$\Rightarrow$ A is singular.
If none of the eigenvalues is zero, then $\lambda = 0$ cannot be a solution to $\det(\lambda I - A) = 0$
$\Rightarrow$ $\det A \neq 0$ $\Rightarrow$ A is invertible. The contrapositive of this statement follows:
A is not invertible $\Rightarrow$ $\det A = 0$ $\Rightarrow$ at least one eigenvalue is zero.

[In logic, the contrapositive of the statement $p \Rightarrow q$ is not $q \Rightarrow$ not p.
If a statement is true, then its contrapositive is true.
If a statement is false, then its contrapositive is false.]
