### 4.1   Vectors and Lines

Scalar quantities are defined by just one number.   Examples of scalars are time, length, electric charge, speed, mass and density.

If a quantity requires information on magnitude **and** direction, then it is a **vector**. Examples of vectors are displacement, velocity, magnetic intensity and force.

Two vectors are equal if and only if they have equal magnitudes and the same directions.

Example 4.1.1

Points A, B, C, D are in the xy-plane as shown.

$\overrightarrow{AB}$ is the **geometric vector** from A to B.

A is the **tail** and B is the **tip** of vector $\overrightarrow{AB}$.

$\overrightarrow{CD}$ is the **geometric vector** from C to D.

But both geometric vectors represent the same displacement (3 units left and 4 up).
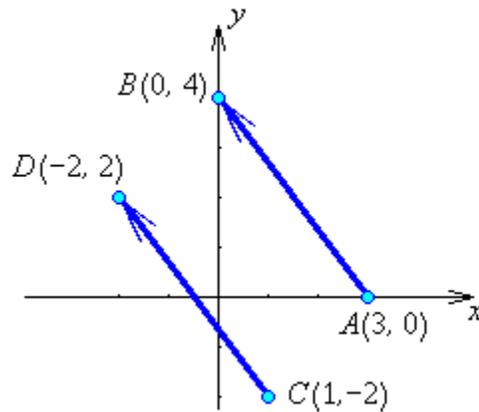
They are both representations of the same underlying vector $\bar{\mathbf{a}} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

The **length** (or magnitude or norm) of a vector $\bar{\mathbf{a}}$ is $\|\bar{\mathbf{a}}\|$.

In this example, $\overrightarrow{AB} = \overrightarrow{CD} = \bar{\mathbf{a}}$ and

$$\|\bar{\mathbf{a}}\| = \|\overrightarrow{AB}\| = \|\overrightarrow{CD}\| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5.$$

A geometric vector may be placed anywhere in space.

The **position vector** of point $P\ (x,\ y,\ z)$ is $\overrightarrow{OP} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

where O is the origin $(0,0,0)$.

Some properties of vectors

Equality:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$ if and only if ($x_1 = x_2$ and $y_1 = y_2$ and $z_1 = z_2$);

Length (or "norm"):

$$\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \Rightarrow \quad \|\vec{\mathbf{v}}\| = \sqrt{x^2 + y^2 + z^2}$$

$\vec{\mathbf{v}} = \vec{\mathbf{0}} \;\Leftrightarrow\; \|\vec{\mathbf{v}}\| = 0$ (and the direction of $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ is undefined)

If $\|\vec{\mathbf{u}}\| = 1$ then $\vec{\mathbf{u}}$ is a **unit vector** (often written as $\hat{\mathbf{u}}$)

The unique unit vector in the same direction as a non-zero vector $\vec{\mathbf{v}}$ is $\hat{\mathbf{v}} = \dfrac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}$.

For any scalar a and any vector **v**, $\|a\vec{\mathbf{v}}\| = |a|\,\|\vec{\mathbf{v}}\|$

In particular, $\|-\vec{\mathbf{v}}\| = \|(-1)\vec{\mathbf{v}}\| = |-1|\|+\vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\|$

Two non-zero vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ are parallel iff one of them is a non-zero scalar multiple a of the other. They point in the same direction if $a > 0$, in opposite directions if $a < 0$.

Parallelogram Law

Vectors can be added "tip to tail":
$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

Consider the vectors in this parallelogram.
$\overrightarrow{BC}$ has the same length and is in the same direction as $\vec{\mathbf{v}} = \overrightarrow{AD}$. Therefore $\overrightarrow{BC} = \vec{\mathbf{v}}$.
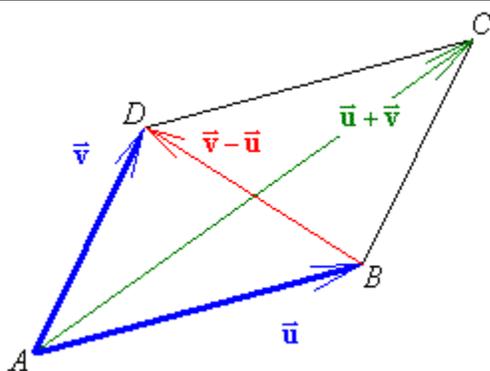
The diagonal vector $\overrightarrow{AC}$ that shares the same tail as both vectors $\vec{\mathbf{u}} = \overrightarrow{AB}$ and $\vec{\mathbf{v}} = \overrightarrow{AD}$ is clearly the sum $\vec{\mathbf{u}} + \vec{\mathbf{v}} = \overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

The other diagonal is $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -\overrightarrow{AB} + \overrightarrow{AD} = -\vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{v}} - \vec{\mathbf{u}}$

The diagram also illustrates the fact that vector addition is commutative:
$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{BC} + \overrightarrow{AB} \quad \Rightarrow \quad \vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{v}} + \vec{\mathbf{u}}$

For any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, $\overrightarrow{P_1P_2} = \begin{bmatrix} (x_2 - x_1) \\ (y_2 - y_1) \\ (z_2 - z_1) \end{bmatrix}$

and $d = \left\| \overrightarrow{P_1P_2} \right\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example 4.1.2

Find the vector from point $A(5, 1, -4)$ to point $B(-3, 10, 8)$, find the distance d between these two points, find the unit vector $\hat{v}$ in the direction $\overrightarrow{AB}$ and find the coordinates of the point P three quarters of the way from A to B.

$\overrightarrow{AB} = \begin{bmatrix} (-3-5) \\ (10-1) \\ (8+4) \end{bmatrix} = \begin{bmatrix} -8 \\ 9 \\ 12 \end{bmatrix}$ and

$d = \left\| \overrightarrow{AB} \right\| = \sqrt{(-8)^2 + 9^2 + 12^2} = \sqrt{64 + 81 + 144} = \sqrt{289}$

$\Rightarrow d = 17$

$\hat{v} = \dfrac{\overrightarrow{AB}}{\left\| \overrightarrow{AB} \right\|} = \dfrac{1}{17}\begin{bmatrix} -8 & 9 & 12 \end{bmatrix}^{\mathrm{T}}$

$\overrightarrow{AP} = \dfrac{3}{4}\overrightarrow{AB} = \dfrac{3}{4}\begin{bmatrix} -8 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} -6 \\ \frac{27}{4} \\ 9 \end{bmatrix}$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix} + \begin{bmatrix} -6 \\ \frac{27}{4} \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{31}{4} \\ 5 \end{bmatrix}$

Therefore P is at $(-1, 7.75, 5)$.

More generally, if we wish to find a point P that divides a line segment AB in the ratio $r : s$, we can proceed as follows.

$$\frac{AP}{PB} = \frac{r}{s} \quad \text{and} \quad AP + PB = AB$$

$$\Rightarrow \quad (AP)s = r(PB) = r(AB - AP)$$

$$\Rightarrow \quad (AP)(s+r) = r(AB) \qquad \Rightarrow \quad AP = \frac{r}{r+s}(AB)$$
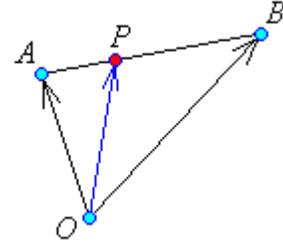
Vectors $\overrightarrow{AP}, \overrightarrow{AB}$ are parallel

$$\Rightarrow \quad \overrightarrow{AP} = \left(\frac{r}{r+s}\right)\overrightarrow{AB}$$

But $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$ and $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$$\Rightarrow \quad \overrightarrow{OP} = \overrightarrow{OA} + \left(\frac{r}{r+s}\right)\left(\overrightarrow{OB} - \overrightarrow{OA}\right) = \left(\frac{r}{r+s}\right)\overrightarrow{OB} + \left(\frac{r+s-r}{r+s}\right)\overrightarrow{OA}$$

Therefore the position vector of P is

$$\boxed{\overrightarrow{OP} = \left(\frac{s}{r+s}\right)\overrightarrow{OA} + \left(\frac{r}{r+s}\right)\overrightarrow{OB}}$$

At the **midpoint** M : $r = s = 1$ and $\boxed{\overrightarrow{OP} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)}$

In Example 4.1.2, $\overrightarrow{OA} = \begin{bmatrix} 5 & 1 & -4 \end{bmatrix}^{T}$, $\overrightarrow{OB} = \begin{bmatrix} -3 & 10 & 8 \end{bmatrix}^{T}$, $r = 3$ and $s = 1$

$$\Rightarrow \quad \overrightarrow{OP} = \left(\frac{3}{4}\right)\begin{bmatrix} -3 \\ 10 \\ 8 \end{bmatrix} + \left(\frac{1}{4}\right)\begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix} = \left(\frac{1}{4}\right)\begin{bmatrix} -4 \\ 31 \\ 16 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{31}{4} \\ 4 \end{bmatrix}$$

Vector Equation of a Line

A line needs two vectors to define it: the position of a point known to be on the line and the orientation of the line in space.

Let $\vec{\mathbf{d}}$ be a vector parallel to the line L.
[$\vec{\mathbf{d}}$ must be a non-zero vector.]

To get from the origin O to a general point P on the line, we follow vectors via a point P_o known to be on the line.

$$\overrightarrow{OP} = \overrightarrow{OP_o} + \overrightarrow{P_oP}$$
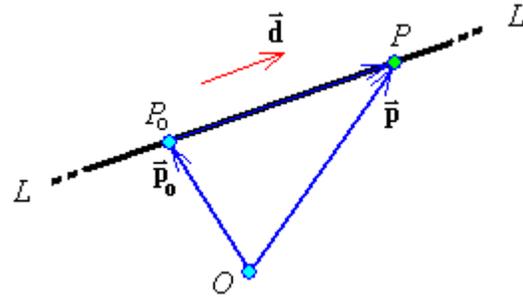
But any vector along the line L is parallel to $\vec{\mathbf{d}}$

$\Rightarrow \quad \overrightarrow{P_oP} = t\,\vec{\mathbf{d}}$, for some value of the scalar t.

As the value of the parameter t varies along the entire range of the real numbers, so the point P sweeps along the entire length of the line L.

The vector parametric equation of the line is therefore

$$\boxed{\vec{\mathbf{p}} = \vec{\mathbf{p}}_{\mathbf{o}} + t\vec{\mathbf{d}}, \quad (t \in \mathbb{R})}$$

where $\vec{\mathbf{p}}_{\mathbf{o}}$ is the position vector of a point P_o known to be on the line and $\vec{\mathbf{d}}$ is a non-zero vector parallel to the line.

This single vector equation generates three Cartesian equations.

The line through $P_o\left(x_o, y_o, z_o\right)$ with direction vector $\vec{\mathbf{d}} = \begin{bmatrix} a & b & c \end{bmatrix}^{\mathrm{T}} \neq \vec{\mathbf{0}}$ is also defined by

$$\boxed{\begin{aligned} x &= x_o + ta \\ y &= y_o + tb, \quad (t \in \mathbb{R}) \\ z &= z_o + tc \end{aligned}}$$

A point $P(x, y, z)$ is on this line if and only if a scalar t exists that satisfies all three equations simultaneously.

Example 4.1.3

Find the equations of the line that passes through the points $A(1,3,5)$ and $B(2,6,9)$.

Either of the points A or B may be taken as the point P_o that is known to be on the line.
Set $\bar{\mathbf{p}}_o = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}^T$.

Vector $\overrightarrow{AB}$ is certainly parallel to the line.

Set $\bar{\mathbf{d}} = \overrightarrow{AB} = \begin{bmatrix} (2-1) \\ (6-3) \\ (9-5) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

The vector parametric equation of the line is

$$\bar{\mathbf{p}} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \quad (t \in \mathbb{R}).$$

The corresponding Cartesian parametric equations are

$$x = 1 + t, \quad y = 3 + 3t, \quad z = 5 + 4t.$$

By making t the subject of all three equations, we arrive at the Cartesian symmetric form of the equations of the line:

$$t = \boxed{\frac{x-1}{1} = \frac{y-3}{3} = \frac{z-5}{4}}$$

Note how the elements of the line direction vector are in the denominators and the coordinates of a point on the line are in the numerator.

Example 4.1.4

Find the points of intersection (if any) of the pair of lines

$$\begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} 5 & 2 & 2 \end{bmatrix}^T + s \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^T$$
and
$$\begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} 6 & -2 & 8 \end{bmatrix}^T + t \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^T$$

At any point of intersection the x coordinates on the two lines must be the same:
$$x = 5 + 2s = 6 + t$$
and the y coordinates on the two lines must be the same:
$$y = 2 + s = -2 - t$$
and the z coordinates on the two lines must be the same:
$$z = 2 = 8 + 2t$$

The last equation yields $t = (2 - 8)/2 = -3$
Substituting into the other two equations:
$$5 + 2s = 6 - 3 \implies 2s = -2 \implies s = -1$$
and
$$2 + s = -2 + 3 \implies s = -1$$

(*or* one may reduce the linear system $\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 1 & -4 \\ 0 & 2 & -6 \end{array} \right]$ to row-echelon form.)

The system of equations is consistent, with the unique solution $s = -1, t = -3$.
$$\implies \quad x = 5 + 2s = 5 - 2 = 3 \,,$$
$$y = 2 + s = 2 - 1 = 1 \,,$$
$$z = 2$$

The unique point of intersection is $(x, y, z) = \underline{(3, 1, 2)}$.

Example 4.1.5

Find the points of intersection (if any) of the pair of lines

$x = 3 + s, \quad y = 1 + s, \quad z = 1 + s$
and
$x = 2 - t, \quad y = 1 - t, \quad z = t$

Matching the coordinates at any point of intersection:

$x \ = \ 3 + s \ = \ 2 - t$
$y \ = \ 1 + s \ = \ 1 - t$
$z \ = \ 1 + s \ = \ t$

From the latter two equations,
$1 - t \ = \ t \ \Rightarrow \ 2t \ = \ 1 \ \Rightarrow \ t = 1/2 \ \Rightarrow \ s = -1/2$

$\Rightarrow \ x \ = \ 3 + s \ = \ 3 - 1/2 \ = \ 5/2 \ $ and $ \ x \ = \ 2 - t \ = \ 2 - 1/2 \ = \ 3/2$
which is inconsistent.

The lines therefore do not meet anywhere.

Or use row operations to carry $\left[\begin{array}{rr|r} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{array}\right]$ to the row-echelon form $\left[\begin{array}{rr|r} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$,

which is inconsistent.

The lines are *not* parallel (their direction vectors are $[\ 1 \ \ 1 \ \ 1 \ ]^{\mathrm{T}}$ and $[\ -1 \ -1 \ \ 1 \ ]^{\mathrm{T}}$).
They are **skew**.

Example 4.1.6   (Textbook, exercises 4.1, pages 167-168, question 34)

The line from a vertex of a triangle to the midpoint of the opposite side is called a **median** of the triangle.   If the vertices of a triangle have position vectors $\bar{\mathbf{u}}, \bar{\mathbf{v}}$ and $\bar{\mathbf{w}}$, show that the point on each median that is a third of the way from the midpoint to the vertex has position vector $\frac{1}{3}(\bar{\mathbf{u}} + \bar{\mathbf{v}} + \bar{\mathbf{w}})$.   Conclude that the point C with position vector $\frac{1}{3}(\bar{\mathbf{u}} + \bar{\mathbf{v}} + \bar{\mathbf{w}})$ lies on all three medians.   This point C is called the **centroid** of the triangle.

$$\overrightarrow{OM} = \tfrac{1}{2}\left(\overrightarrow{OU} + \overrightarrow{OW}\right) = \tfrac{1}{2}(\bar{\mathbf{u}} + \bar{\mathbf{w}})$$

C is one third of the way from M to V
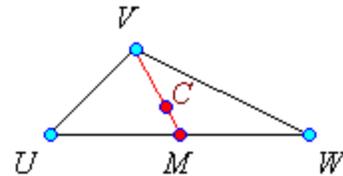
$$\Rightarrow \quad \overrightarrow{OC} = \frac{2}{3}\overrightarrow{OM} + \frac{1}{3}\overrightarrow{OV} = \frac{2}{3}\left(\frac{\bar{\mathbf{u}} + \bar{\mathbf{w}}}{2}\right) + \frac{1}{3}\bar{\mathbf{v}} = \frac{1}{3}(\bar{\mathbf{u}} + \bar{\mathbf{v}} + \bar{\mathbf{w}})$$

A corresponding result holds for the other two medians.

For example, the position vector of the point one third of the way from the midpoint of side VW to vertex U is $\dfrac{2}{3}\left(\dfrac{\bar{\mathbf{v}} + \bar{\mathbf{w}}}{2}\right) + \dfrac{1}{3}\bar{\mathbf{u}} = \dfrac{1}{3}(\bar{\mathbf{u}} + \bar{\mathbf{v}} + \bar{\mathbf{w}})$.

Therefore the point C lies on all three medians.

4.2   Projections and Planes

In section 2.2 matrix multiplication of an ($m \times p$) matrix A by a ($p \times n$) matrix B was introduced as an array of dot products of row vectors with column vectors:

$$C = AB = \begin{bmatrix} c_{ij} \end{bmatrix}, \quad c_{ij} = R_i C_j = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{ip} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

The **dot product** of two vectors $\bar{\mathbf{u}} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\bar{\mathbf{v}} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ is similarly defined as

$$\bar{\mathbf{u}} \bullet \bar{\mathbf{v}} = \bar{\mathbf{u}}^T \bar{\mathbf{v}} = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Because the dot product of two vectors is a number (a scalar), it is also known as the **scalar product**.

Example 4.2.1

$$\bar{\mathbf{u}} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}, \quad \bar{\mathbf{v}} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \bar{\mathbf{u}} \bullet \bar{\mathbf{v}} = \bar{\mathbf{u}}^T \bar{\mathbf{v}} = 4 \times (-1) + (-2) \times 2 + 3 \times 2 = -2$$

Properties of the dot product

Let $\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\mathbf{w}}$ denote vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and k be any scalar. Then

$\bar{\mathbf{u}} \bullet \bar{\mathbf{v}} \in \mathbb{R}$ (the dot product is a real number)

$\bar{\mathbf{u}} \bullet \bar{\mathbf{v}} = \bar{\mathbf{v}} \bullet \bar{\mathbf{u}}$ (the dot product is commutative)

$\bar{\mathbf{0}} \bullet \bar{\mathbf{v}} = 0 = \bar{\mathbf{v}} \bullet \bar{\mathbf{0}}$ (zero vector)

$\bar{\mathbf{v}} \bullet \bar{\mathbf{v}} = \left\| \bar{\mathbf{v}} \right\|^2$ (length^2)

$\left(k \bar{\mathbf{u}} \right) \bullet \bar{\mathbf{v}} = k \left(\bar{\mathbf{u}} \bullet \bar{\mathbf{v}} \right) = \bar{\mathbf{u}} \bullet \left(k \bar{\mathbf{v}} \right)$ (scalar multiplication)

$\bar{\mathbf{u}} \bullet \left(\bar{\mathbf{v}} \pm \bar{\mathbf{w}} \right) = \bar{\mathbf{u}} \bullet \bar{\mathbf{v}} \pm \bar{\mathbf{u}} \bullet \bar{\mathbf{w}}$ (the dot product is distributive over addition)

Angle between Two Vectors

Let θ be the angle between vectors $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$.

Apply the law of cosines to the triangle.

$$(BC)^2 = (AB)^2 + (AC)^2 - 2(AB)(AC)\cos\theta$$

$$\Rightarrow \ \| \bar{\mathbf{v}} - \bar{\mathbf{u}} \|^2 = \| \bar{\mathbf{u}} \|^2 + \| \bar{\mathbf{v}} \|^2 - 2\| \bar{\mathbf{u}} \|\| \bar{\mathbf{v}} \|\cos\theta$$

But $\| \bar{\mathbf{v}} - \bar{\mathbf{u}} \|^2 = (\bar{\mathbf{v}} - \bar{\mathbf{u}}) \bullet (\bar{\mathbf{v}} - \bar{\mathbf{u}}) = \bar{\mathbf{v}} \bullet \bar{\mathbf{v}} - \bar{\mathbf{u}} \bullet \bar{\mathbf{v}} - \bar{\mathbf{v}} \bullet \bar{\mathbf{u}} + \bar{\mathbf{u}} \bullet \bar{\mathbf{u}}$

$$= \| \bar{\mathbf{v}} \|^2 - 2\bar{\mathbf{u}} \bullet \bar{\mathbf{v}} + \| \bar{\mathbf{u}} \|^2$$

$$\Rightarrow \ -2\| \bar{\mathbf{u}} \|\| \bar{\mathbf{v}} \|\cos\theta = -2\bar{\mathbf{u}} \bullet \bar{\mathbf{v}} \quad \Rightarrow$$

$$\boxed{\bar{\mathbf{u}} \bullet \bar{\mathbf{v}} = \| \bar{\mathbf{u}} \|\| \bar{\mathbf{v}} \|\cos\theta}$$

and the angle θ between non-zero vectors $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ can be found from

$$\boxed{\cos\theta = \frac{\bar{\mathbf{u}} \bullet \bar{\mathbf{v}}}{\| \bar{\mathbf{u}} \|\| \bar{\mathbf{v}} \|}}$$

Recall that, for an acute angle θ (first quadrant),
$0 < \theta < \frac{\pi}{2} \ \Rightarrow \ \cos\theta > 0$

and for an obtuse angle (second quadrant),
$\frac{\pi}{2} < \theta < \pi \ \Rightarrow \ \cos\theta < 0$

If the dot product is negative, then
the angle between the two vectors is obtuse.
The vectors are pointing away (approximately) from each other.

If two non-zero vectors are at right angles, then $\bar{\mathbf{u}} \bullet \bar{\mathbf{v}} = 0$.

If $\bar{\mathbf{u}} \bullet \bar{\mathbf{v}} = 0$ then $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ are **orthogonal** and
either $\bar{\mathbf{u}} = \bar{\mathbf{0}}$ or $\bar{\mathbf{v}} = \bar{\mathbf{0}}$ or the angle between the vectors is $\frac{\pi}{2}$.
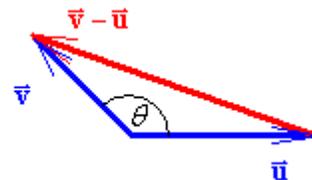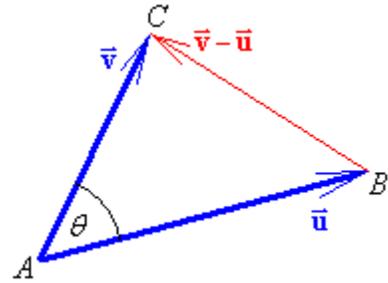
<u>Example 4.2.2</u>

Points $A(3,0,1)$, $B(5,1,0)$ and $C(9,2,2)$ define a triangle ABC.
Show that the triangle contains an obtuse angle.

$$\overrightarrow{AB} \;=\; \begin{bmatrix} 5 & 1 & 0 \end{bmatrix}^{\mathrm{T}} - \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}^{\mathrm{T}} \;=\; \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\overrightarrow{AC} \;=\; \begin{bmatrix} 9 & 2 & 2 \end{bmatrix}^{\mathrm{T}} - \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}^{\mathrm{T}} \;=\; \begin{bmatrix} 6 & 2 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\overrightarrow{BC} \;=\; \begin{bmatrix} 9 & 2 & 2 \end{bmatrix}^{\mathrm{T}} - \begin{bmatrix} 5 & 1 & 0 \end{bmatrix}^{\mathrm{T}} \;=\; \begin{bmatrix} 4 & 1 & 2 \end{bmatrix}^{\mathrm{T}}$$

The vectors with tails at point B are $\bar{\mathbf{u}} = \overrightarrow{BC}$ and $\bar{\mathbf{v}} = \overrightarrow{BA}$

$$\bar{\mathbf{u}}{\boldsymbol\cdot}\bar{\mathbf{v}} = \begin{bmatrix} 4 & 1 & 2 \end{bmatrix}^{\mathrm{T}}{\boldsymbol\cdot}\begin{bmatrix} -2 & -1 & 1 \end{bmatrix}^{\mathrm{T}} \;=\; 4\times(-2)+1\times(-1)+2\times1 \;=\; -8-1+2 \;=\; -7<0$$

$\overrightarrow{BC}{\boldsymbol\cdot}\overrightarrow{BA} < 0 \qquad \Rightarrow \quad$ the angle at B is obtuse.
[One can quickly deduce that $\overrightarrow{AB}{\boldsymbol\cdot}\overrightarrow{AC} > 0$ and $\overrightarrow{CB}{\boldsymbol\cdot}\overrightarrow{CA} > 0$,
so that the angles at A and at C are both acute.]

<u>Example 4.2.3</u>

Find all real numbers x such that $\begin{bmatrix} x & 1 & 2 \end{bmatrix}^{\mathrm{T}}$ and $\begin{bmatrix} x & -3 & -x \end{bmatrix}^{\mathrm{T}}$ are orthogonal.

$$\begin{bmatrix} x \\ 1 \\ 2 \end{bmatrix}{\boldsymbol\cdot}\begin{bmatrix} x \\ -3 \\ x \end{bmatrix} \;=\; x^2 - 3 + 2x \;=\; (x+3)(x-1)$$

For orthogonality, $(x+3)(x-1)=0 \quad \Rightarrow \quad \underline{x = -3 \text{ or } x = 1}$.

One can quickly check that $\begin{bmatrix} -3 & 1 & 2 \end{bmatrix}^{\mathrm{T}}$ and $\begin{bmatrix} -3 & -3 & -3 \end{bmatrix}^{\mathrm{T}}$ are orthogonal and that
$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix}^{\mathrm{T}}$ and $\begin{bmatrix} 1 & -3 & 1 \end{bmatrix}^{\mathrm{T}}$ are orthogonal.

Projections

In this diagram, $\vec{\mathbf{u}} = \overrightarrow{QP_1}$ is the shadow of vector $\vec{\mathbf{v}} = \overrightarrow{QP}$ on the line L that passes through Q and whose line direction vector is the non-zero vector $\vec{\mathbf{d}}$.
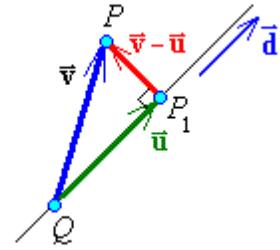
$\vec{\mathbf{u}} = \overrightarrow{QP_1}$ is parallel to $\vec{\mathbf{d}}$.

$\vec{\mathbf{v}} - \vec{\mathbf{u}} = \overrightarrow{P_1 P}$ is orthogonal to $\vec{\mathbf{d}}$.

The vector $\vec{\mathbf{v}} = \overrightarrow{QP}$ has therefore been decomposed into a pair of orthogonal vectors, one parallel to the line and the other orthogonal to the line.

The "shadow" vector $\vec{\mathbf{u}} = \overrightarrow{QP_1}$ is the **projection** of $\vec{\mathbf{v}}$ on $\vec{\mathbf{d}}$, denoted by $\vec{\mathbf{u}} = \text{proj}_{\vec{\mathbf{d}}} \vec{\mathbf{v}}$.

$\vec{\mathbf{u}} \parallel \vec{\mathbf{d}} \implies \vec{\mathbf{u}} = t\vec{\mathbf{d}}$ for some scalar t.

$$\left(\vec{\mathbf{v}} - \vec{\mathbf{u}} \right) \perp \vec{\mathbf{d}} \implies \left(\vec{\mathbf{v}} - t\vec{\mathbf{d}} \right) \cdot \vec{\mathbf{d}} = 0 \implies \vec{\mathbf{v}} \cdot \vec{\mathbf{d}} - t\vec{\mathbf{d}} \cdot \vec{\mathbf{d}} = 0 \implies \vec{\mathbf{v}} \cdot \vec{\mathbf{d}} = t \left\| \vec{\mathbf{d}} \right\|^2$$

$$\implies t = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{d}}}{\left\| \vec{\mathbf{d}} \right\|^2} \implies \vec{\mathbf{u}} = \left(\frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{d}}}{\left\| \vec{\mathbf{d}} \right\|^2} \right) \vec{\mathbf{d}}, \text{ (which requires } \vec{\mathbf{d}} \neq \vec{\mathbf{0}}).$$

Another way of writing the projection, in terms of the unit vector $\hat{\mathbf{d}}$ parallel to the line, is

$$\boxed{\text{proj}_{\vec{\mathbf{d}}} \vec{\mathbf{v}} = \left(\vec{\mathbf{v}} \cdot \hat{\mathbf{d}} \right) \hat{\mathbf{d}}}$$

and the vector $\vec{\mathbf{v}} - \text{proj}_{\vec{\mathbf{d}}} \vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{d}}$.

Example 4.2.4

Express the vector $\bar{\mathbf{v}} = \begin{bmatrix} 3 & 4 & 2 \end{bmatrix}^{\mathrm{T}}$ as $\bar{\mathbf{v}} = \bar{\mathbf{a}} + \bar{\mathbf{b}}$, where $\bar{\mathbf{a}}$ is parallel to $\bar{\mathbf{d}} = \begin{bmatrix} 4 & 4 & 7 \end{bmatrix}^{\mathrm{T}}$ and $\bar{\mathbf{b}}$ is orthogonal to $\bar{\mathbf{d}}$.

$$\bar{\mathbf{d}} = \begin{bmatrix} 4 & 4 & 7 \end{bmatrix}^{\mathrm{T}} \quad \Rightarrow \quad \|\bar{\mathbf{d}}\| = \sqrt{16+16+49} = \sqrt{81} = 9$$

$$\Rightarrow \quad \hat{\mathbf{d}} = \frac{\bar{\mathbf{d}}}{\|\bar{\mathbf{d}}\|} = \frac{\begin{bmatrix} 4 & 4 & 7 \end{bmatrix}^{\mathrm{T}}}{9}$$

$$\bar{\mathbf{a}} = \mathrm{proj}_{\bar{\mathbf{d}}}\,\bar{\mathbf{v}} = \left(\bar{\mathbf{v}}\cdot\hat{\mathbf{d}}\right)\hat{\mathbf{d}} = \left(\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}\cdot\frac{1}{9}\begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix}\right)\hat{\mathbf{d}} = \frac{1}{9}(12+16+14)\frac{1}{9}\begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix} = \frac{42}{81}\begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix}$$

$$\Rightarrow \quad \bar{\mathbf{a}} = \frac{14}{27}\begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix}$$

$$\bar{\mathbf{b}} = \bar{\mathbf{v}} - \bar{\mathbf{a}} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} - \frac{14}{27}\begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} (81-56) \\ (108-56) \\ (54-98) \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 25 \\ 52 \\ -44 \end{bmatrix}$$

$$\text{and } \bar{\mathbf{v}} = \bar{\mathbf{a}} + \bar{\mathbf{b}} = \frac{14}{27}\begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix} + \frac{1}{27}\begin{bmatrix} 25 \\ 52 \\ -44 \end{bmatrix}$$

$\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ should be orthogonal.   Checking our answers:

$$\bar{\mathbf{a}}\cdot\bar{\mathbf{b}} = \frac{14}{27}\begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix}\cdot\frac{1}{27}\begin{bmatrix} 25 \\ 52 \\ -44 \end{bmatrix} = \frac{14}{27^2}(100+208-308) = 0 \quad \checkmark$$

Distance of a Point from a Line

The distance r of a point P from the line through Q
with direction vector $\bar{\mathbf{d}}$ is clearly $r = PP_1 = \left\| \bar{\mathbf{v}} - \text{proj}_{\bar{\mathbf{d}}} \bar{\mathbf{v}} \right\|$,

where $\bar{\mathbf{v}} = \overrightarrow{QP}$.
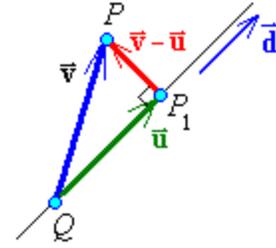
Example 4.2.5

Find the distance from the point $P(3, -1, 4)$ to the line through the points $Q(0, 1, 3)$ and $R(2, 2, 7)$.

The line direction vector is $\bar{\mathbf{d}} = \overrightarrow{QR} = \begin{bmatrix} 2 & 2 & 7 \end{bmatrix}^T - \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & 4 \end{bmatrix}^T$

$\Rightarrow \left\| \bar{\mathbf{d}} \right\| = \sqrt{4+1+16} = \sqrt{21} \qquad \Rightarrow \hat{\mathbf{d}} = \dfrac{\bar{\mathbf{d}}}{\left\| \bar{\mathbf{d}} \right\|} = \dfrac{1}{\sqrt{21}} \begin{bmatrix} 2 & 1 & 4 \end{bmatrix}^T$

$\bar{\mathbf{v}} = \overrightarrow{QP} = \begin{bmatrix} 3 & -1 & 4 \end{bmatrix}^T - \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}^T$

$\text{proj}_{\bar{\mathbf{d}}} \bar{\mathbf{v}} = \left(\bar{\mathbf{v}} \cdot \hat{\mathbf{d}} \right) \hat{\mathbf{d}} = \left(\begin{bmatrix} 3 & -2 & 1 \end{bmatrix}^T \cdot \dfrac{\begin{bmatrix} 2 & 1 & 4 \end{bmatrix}^T}{\sqrt{21}} \right) \hat{\mathbf{d}}$

$= \left(\dfrac{6-2+4}{\sqrt{21}} \right) \hat{\mathbf{d}} = \left(\dfrac{8}{\sqrt{21}} \right) \dfrac{\begin{bmatrix} 2 & 1 & 4 \end{bmatrix}^T}{\sqrt{21}} = \dfrac{8}{21} \begin{bmatrix} 2 & 1 & 4 \end{bmatrix}^T$

$\Rightarrow \overrightarrow{P_1 P} = \bar{\mathbf{v}} - \text{proj}_{\bar{\mathbf{d}}} \bar{\mathbf{v}} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}^T - \dfrac{8}{21} \begin{bmatrix} 2 & 1 & 4 \end{bmatrix}^T$

$= \dfrac{1}{21} \left(\begin{bmatrix} (63-16) & (-42-8) & (21-32) \end{bmatrix}^T \right) = \dfrac{1}{21} \begin{bmatrix} 47 & -50 & -11 \end{bmatrix}^T$

$\Rightarrow r = \left\| \bar{\mathbf{v}} - \text{proj}_{\bar{\mathbf{d}}} \bar{\mathbf{v}} \right\| = \dfrac{1}{21} \sqrt{2209+2500+121} = \dfrac{\sqrt{4830}}{21} = \sqrt{\dfrac{230}{21}} \approx 3.31$

Also the location of the nearest point P_1 on the line to the point P can be found:

$\overrightarrow{OP_1} = \overrightarrow{OP} + \overrightarrow{PP_1} = \begin{bmatrix} 3 & -1 & 4 \end{bmatrix}^T - \dfrac{1}{21} \begin{bmatrix} 47 & -50 & -11 \end{bmatrix}^T \Rightarrow P_1 \text{ is at } \left(\tfrac{16}{21}, \tfrac{29}{21}, \tfrac{95}{21} \right).$

Equations of Planes

A plane, like a line, requires two vectors to define it: one vector for its orientation in space, the other $(\bar{\mathbf{p}}_0)$ to fix the location of any one point P_o known to be on the plane.

Unlike a line, the vector that defines a plane in $\mathbb{R}^3$ is its **normal $\bar{\mathbf{n}}$** (any non-zero vector that is orthogonal to all vectors lying in or parallel to the plane).
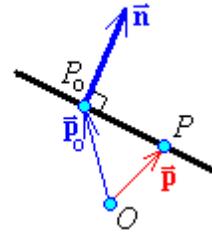
If and only if the point P lies in the plane, then the vector $\overrightarrow{P_o P}$ is orthogonal to the plane's normal vector $\bar{\mathbf{n}}$. But $\overrightarrow{P_o P} = \overrightarrow{P_o O} + \overrightarrow{OP} = \bar{\mathbf{p}} - \bar{\mathbf{p}}_0$.

Therefore the vector equation of the plane is

$$\boxed{\bar{\mathbf{n}} \cdot (\bar{\mathbf{p}} - \bar{\mathbf{p}}_0) = 0}$$

Another way to look at the equation of a plane is to note that the projections of vectors $\overrightarrow{OP}$ and $\overrightarrow{OP_o}$ in the direction of the normal vector $\bar{\mathbf{n}}$ will be equal if and only if point P is on the plane. The equation of the plane is then

$$\left(\overrightarrow{OP} \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}} = \left(\overrightarrow{OP_o} \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}} \quad \Rightarrow \quad \mathbf{p} \cdot \hat{\mathbf{n}} = \mathbf{p_0} \cdot \hat{\mathbf{n}} \quad \Rightarrow \quad \hat{\mathbf{n}} \cdot (\mathbf{p} - \mathbf{p_0}) = 0 \quad \Rightarrow \quad \bar{\mathbf{n}} \cdot (\mathbf{p} - \mathbf{p_0}) = 0$$

If the normal vector to the plane is $\bar{\mathbf{n}} = \begin{bmatrix} a & b & c \end{bmatrix}^T \neq \vec{\mathbf{0}}$, then the equation of the plane becomes $\begin{bmatrix} a & b & c \end{bmatrix}^T \cdot \left(\begin{bmatrix} x & y & z \end{bmatrix}^T - \begin{bmatrix} x_o & y_o & z_o \end{bmatrix}^T \right) = 0$

$\Rightarrow a(x - x_o) + b(y - y_o) + c(z - z_o) = 0$

or, defining a new constant d as $d = -(ax_o + by_o + cz_o) = -\bar{\mathbf{n}} \cdot \bar{\mathbf{p}}_0$, the equation of a plane with normal vector $\bar{\mathbf{n}} = \begin{bmatrix} a & b & c \end{bmatrix}^T$ is

$$\boxed{ax + by + cz + d = 0}$$

Example 4.2.6

Find the equation of the plane that is orthogonal to the line $\dfrac{x-2}{1} = \dfrac{y-(-3)}{3} = \dfrac{z-0}{4}$ and passes through the point (1, 1, 1).

Any normal vector to the plane is parallel to the direction vector $\vec{d}$ of the line.

Therefore let $\bar{\mathbf{n}} = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix}^T$.

The plane passes through the point (1, 1, 1)

$\Rightarrow \bar{\mathbf{p}}_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$

$\Rightarrow \bar{\mathbf{n}} \cdot \bar{\mathbf{p}}_0 = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix}^T \cdot \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T = 1+3+4 = 8$

Also note that

$\bar{\mathbf{n}} \cdot \bar{\mathbf{p}} = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix}^T \cdot \begin{bmatrix} x & y & z \end{bmatrix}^T = 1x + 3y + 4z$

Therefore the equation of the plane is $\bar{\mathbf{n}} \cdot \bar{\mathbf{p}} = \bar{\mathbf{n}} \cdot \bar{\mathbf{p}}_0$ or

$$x + 3y + 4z = 8$$
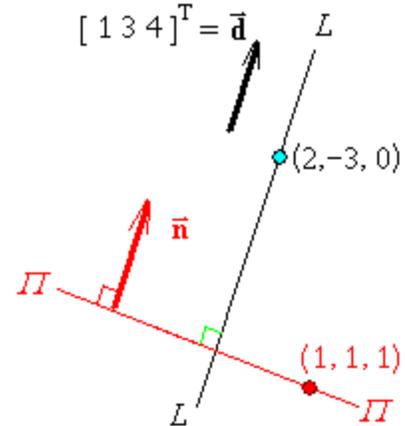
Example 4.2.7

Find the equation of the plane that is parallel to the plane $4x - 3y + 5z = 10$ and passes through the point (3, 7, 2).


The two planes are parallel

$\Rightarrow$ their normal vectors are parallel.

Therefore let $\bar{\mathbf{n}} = \begin{bmatrix} 4 & -3 & 5 \end{bmatrix}^T$.

The plane passes through the point (3, 7, 2)

$\Rightarrow \bar{\mathbf{p}}_0 = \begin{bmatrix} 3 & 7 & 2 \end{bmatrix}^T$

$\Rightarrow \bar{\mathbf{n}} \cdot \bar{\mathbf{p}}_0 = \begin{bmatrix} 4 & -3 & 5 \end{bmatrix}^T \cdot \begin{bmatrix} 3 & 7 & 2 \end{bmatrix}^T = 12-21+10 = 1$

Therefore the equation of the plane is $\bar{\mathbf{n}} \cdot \bar{\mathbf{p}} = \bar{\mathbf{n}} \cdot \bar{\mathbf{p}}_0$ or

$$4x - 3y + 5z = 1$$
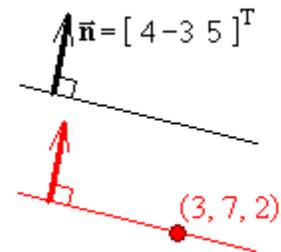
Coordinate Basis Vectors

In Cartesian coordinates in $\mathbb{R}^3$, the basis vectors are

$\hat{\mathbf{i}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ (or just $\mathbf{i}$), a unit vector pointing along the x axis,

$\hat{\mathbf{j}} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ (or $\mathbf{j}$), a unit vector pointing along the y axis and

$\hat{\mathbf{k}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ (or $\mathbf{k}$), a unit vector pointing along the z axis.
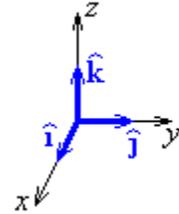
Any position vector $\bar{\mathbf{p}} = \begin{bmatrix} x & y & z \end{bmatrix}^T$ can be written as $\bar{\mathbf{p}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

Cross Product

The normal vector to a plane is orthogonal to all vectors lying in that plane.
If we know two non-parallel non-zero vectors in the plane, then any function of those vectors that results in a non-zero vector at right angles to both of them will provide a normal vector to the plane.   The cross product provides such an orthogonal vector.

For any two vectors $\bar{\mathbf{u}} = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}^T$ and $\bar{\mathbf{v}} = \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix}^T$ the cross product of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\bar{\mathbf{u}} \times \bar{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & x_1 & x_2 \\ \hat{\mathbf{j}} & y_1 & y_2 \\ \hat{\mathbf{k}} & z_1 & z_2 \end{vmatrix}$$

Expanding down column 1,

$$\bar{\mathbf{u}} \times \bar{\mathbf{v}} = \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \hat{\mathbf{k}} = \begin{bmatrix} (y_1 z_2 - y_2 z_1) \\ (z_1 x_2 - z_2 x_1) \\ (x_1 y_2 - x_2 y_1) \end{bmatrix}$$

This vector is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.   Proof for $\mathbf{u}$:

$$\bar{\mathbf{u}} \bullet (\bar{\mathbf{u}} \times \bar{\mathbf{v}}) = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \bullet \begin{bmatrix} (y_1 z_2 - y_2 z_1) \\ (z_1 x_2 - z_2 x_1) \\ (x_1 y_2 - x_2 y_1) \end{bmatrix}$$

$$= \left(x_1 y_1 z_2 - x_1 y_2 z_1 \right) + \left(y_1 z_1 x_2 - y_1 z_2 x_1 \right) + \left(z_1 x_1 y_2 - z_1 x_2 y_1 \right) = 0$$

The proof for $\mathbf{v}$ is similar.

Equation of a Plane using the Cross Product

If P, Q, R are three points in a plane, not all on the same line, then a normal vector to that plane is $\bar{\mathbf{n}} = \overrightarrow{PQ} \times \overrightarrow{PR}$.

Either of the other pairs of vectors can be used instead:
$\bar{\mathbf{n}}_2 = \overrightarrow{QR} \times \overrightarrow{QP}$ or $\bar{\mathbf{n}}_3 = \overrightarrow{RP} \times \overrightarrow{RQ}$.   In general the magnitudes of these three cross product vectors will be different, but they will all be parallel to each other and normal to the plane.
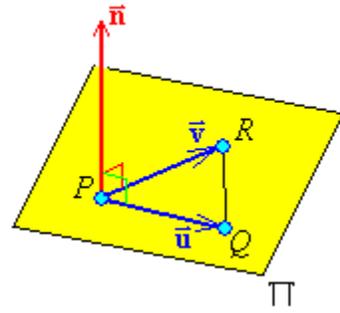
Example 4.2.8

Find the equation of the plane that passes through the points $P(0, 2, 1)$, $Q(3, 2, 4)$ and $R(1, 5, 7)$.

$\overrightarrow{PQ} = \begin{bmatrix} 3 & 0 & 3 \end{bmatrix}^{\mathrm{T}}$ and $\overrightarrow{PR} = \begin{bmatrix} 1 & 3 & 6 \end{bmatrix}^{\mathrm{T}}$

$$\Rightarrow \ \bar{\mathbf{u}} \times \bar{\mathbf{v}} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\mathbf{i}} & 3 & 1 \\ \hat{\mathbf{j}} & 0 & 3 \\ \hat{\mathbf{k}} & 3 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 3 & 6 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 3 & 1 \\ 3 & 6 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 3 & 1 \\ 0 & 3 \end{vmatrix} \hat{\mathbf{k}}$$

$$\Rightarrow \ \bar{\mathbf{u}} \times \bar{\mathbf{v}} = \begin{bmatrix} -9 & -15 & 9 \end{bmatrix}^{\mathrm{T}} = -3 \begin{bmatrix} 3 & 5 & -3 \end{bmatrix}^{\mathrm{T}}$$

Any non-zero multiple of a normal vector is also a normal vector.
Therefore take $\bar{\mathbf{n}} = \begin{bmatrix} 3 & 5 & -3 \end{bmatrix}^{\mathrm{T}}$
For the vector $\bar{\mathbf{p}}_0$ any of $\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}$ could be used. Choosing P,

$$\bar{\mathbf{p}}_0 = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}^{\mathrm{T}} \quad \Rightarrow \quad \bar{\mathbf{n}} \bullet \bar{\mathbf{p}}_0 = \begin{bmatrix} 3 & 5 & -3 \end{bmatrix}^{\mathrm{T}} \bullet \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}^{\mathrm{T}} = 0 + 10 - 3 = 7$$

and $\quad \bar{\mathbf{n}} \bullet \bar{\mathbf{p}} = \begin{bmatrix} 3 & 5 & -3 \end{bmatrix}^{\mathrm{T}} \bullet \begin{bmatrix} x & y & z \end{bmatrix}^{\mathrm{T}} = 3x + 5y - 3z$

Therefore the equation of the plane is $\bar{\mathbf{n}} \bullet \bar{\mathbf{p}} = \bar{\mathbf{n}} \bullet \bar{\mathbf{p}}_0$ or

$$3x + 5y - 3z = 7$$

4.3   __The Cross Product__

Quoting from section 4.2,

For any two vectors $\bar{\mathbf{u}} = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}^T$ and $\bar{\mathbf{v}} = \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix}^T$ the cross product of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$
\bar{\mathbf{u}} \times \bar{\mathbf{v}} \;=\; \begin{vmatrix} \hat{\mathbf{i}} & x_1 & x_2 \\ \hat{\mathbf{j}} & y_1 & y_2 \\ \hat{\mathbf{k}} & z_1 & z_2 \end{vmatrix}
$$

From this it follows that $\bar{\mathbf{v}} \times \bar{\mathbf{u}}$ is obtained from $\bar{\mathbf{u}} \times \bar{\mathbf{v}}$ by interchanging columns 2 and 3 of the determinant. But this interchange introduces a change of sign. Therefore $\bar{\mathbf{v}} \times \bar{\mathbf{u}} = -\bar{\mathbf{u}} \times \bar{\mathbf{v}}$ - the cross product is anti-symmetric (and therefore is __not__ commutative). Setting $\bar{\mathbf{v}}$ = any multiple k of $\bar{\mathbf{u}}$, we have

$$
(k\bar{\mathbf{u}}) \times \bar{\mathbf{u}} = -\bar{\mathbf{u}} \times (k\bar{\mathbf{u}}) \;\;\Rightarrow\;\; 2k(\bar{\mathbf{u}} \times \bar{\mathbf{u}}) = \bar{\mathbf{0}} \;\;\Rightarrow\;\; \bar{\mathbf{u}} \times \bar{\mathbf{u}} = \bar{\mathbf{0}}
$$

Therefore the cross product of any pair of parallel vectors is the zero vector.
Also

$$
\bar{\mathbf{u}} \times \bar{\mathbf{0}} = \bar{\mathbf{0}} = \bar{\mathbf{0}} \times \bar{\mathbf{u}}
$$

$$
\bar{\mathbf{u}} \times (\bar{\mathbf{v}} + \bar{\mathbf{w}}) = (\bar{\mathbf{u}} \times \bar{\mathbf{v}}) + (\bar{\mathbf{u}} \times \bar{\mathbf{w}})
$$

$$
(\bar{\mathbf{v}} + \bar{\mathbf{w}}) \times \bar{\mathbf{u}} = (\bar{\mathbf{v}} \times \bar{\mathbf{u}}) + (\bar{\mathbf{w}} \times \bar{\mathbf{u}})
$$

Lagrange identity:

$$
\left\| \bar{\mathbf{u}} \times \bar{\mathbf{v}} \right\|^2 \;=\; \left\| \bar{\mathbf{u}} \right\|^2 \left\| \bar{\mathbf{v}} \right\|^2 \;-\; (\bar{\mathbf{u}} \bullet \bar{\mathbf{v}})^2
$$

But $\bar{\mathbf{u}} \bullet \bar{\mathbf{v}} = \left\| \bar{\mathbf{u}} \right\| \left\| \bar{\mathbf{v}} \right\| \cos\theta$, where θ is the angle between the two vectors

$$
\Rightarrow\; \left\| \bar{\mathbf{u}} \times \bar{\mathbf{v}} \right\|^2 \;=\; \left\| \bar{\mathbf{u}} \right\|^2 \left\| \bar{\mathbf{v}} \right\|^2 \;-\; \left\| \bar{\mathbf{u}} \right\|^2 \left\| \bar{\mathbf{v}} \right\|^2 (\cos\theta)^2
$$

$$
= \left(\left\| \bar{\mathbf{u}} \right\| \left\| \bar{\mathbf{v}} \right\| \right)^2 (1 - \cos^2\theta) \;=\; \left(\left\| \bar{\mathbf{u}} \right\| \left\| \bar{\mathbf{v}} \right\| \sin\theta \right)^2
$$

Therefore a coordinate-free geometrical interpretation of the cross product is

$$
\left\| \bar{\mathbf{u}} \times \bar{\mathbf{v}} \right\| \;=\; \left\| \bar{\mathbf{u}} \right\| \left\| \bar{\mathbf{v}} \right\| \sin\theta
$$

The **scalar triple product** of vectors $\bar{\mathbf{u}} = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}^T$, $\bar{\mathbf{v}} = \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix}^T$ and

$\bar{\mathbf{w}} = \begin{bmatrix} x_3 & y_3 & z_3 \end{bmatrix}^T$ is $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} \times \bar{\mathbf{w}} = \det \begin{bmatrix} \bar{\mathbf{u}} & \bar{\mathbf{v}} & \bar{\mathbf{w}} \end{bmatrix}$, where $\begin{bmatrix} \bar{\mathbf{u}} & \bar{\mathbf{v}} & \bar{\mathbf{w}} \end{bmatrix}$ is the matrix with $\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\mathbf{w}}$ as its columns.

Note that the cross product has to be evaluated first. The cross product of a scalar with a vector is not defined. Therefore $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} \times \bar{\mathbf{w}} = \bar{\mathbf{u}} \cdot (\bar{\mathbf{v}} \times \bar{\mathbf{w}})$ and $\bar{\mathbf{u}} \times \bar{\mathbf{v}} \cdot \bar{\mathbf{w}} = (\bar{\mathbf{u}} \times \bar{\mathbf{v}}) \cdot \bar{\mathbf{w}}$.

Also

$$\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} \times \bar{\mathbf{w}} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Proof:

Expanding down column 1:

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = +x_1 \begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix} - y_1 \begin{vmatrix} x_2 & x_3 \\ z_2 & z_3 \end{vmatrix} + z_1 \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}$$

$$= \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}^T \cdot \begin{bmatrix} + \begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix} & - \begin{vmatrix} x_2 & x_3 \\ z_2 & z_3 \end{vmatrix} & + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} \end{bmatrix}^T = \bar{\mathbf{u}} \cdot (\bar{\mathbf{v}} \times \bar{\mathbf{w}})$$

Geometric interpretations:

The area of the parallelogram *ABCD* defined by vectors $\bar{\mathbf{u}}, \bar{\mathbf{v}}$ is

$A = (\text{base}) \times (\text{perpendicular height})$

$= \|\bar{\mathbf{u}}\| \left(\|\bar{\mathbf{v}}\| \sin\theta \right)$

Therefore $A = \|\bar{\mathbf{u}} \times \bar{\mathbf{v}}\|$
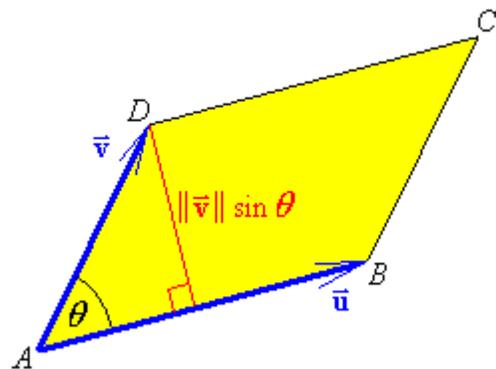
and it also follows that the area of triangle *ABD* is $\frac{1}{2}\|\bar{\mathbf{u}} \times \bar{\mathbf{v}}\|$.

Example 4.3.1

Find the area of the triangle whose vertices are at the points $P(0, 2, 1)$, $Q(3, 2, 4)$ and $R(1, 5, 7)$.

$\overrightarrow{PQ} = \begin{bmatrix} 3 & 0 & 3 \end{bmatrix}^T$, $\overrightarrow{QR} = \begin{bmatrix} -2 & 3 & 3 \end{bmatrix}^T$ and $\overrightarrow{PR} = \begin{bmatrix} 1 & 3 & 6 \end{bmatrix}^T$

The cross product of any two of these three vectors may be used to evaluate the area.

Selecting $\bar{\mathbf{u}} = \overrightarrow{PQ} = \begin{bmatrix} 3 & 0 & 3 \end{bmatrix}^T$ and $\bar{\mathbf{v}} = \overrightarrow{PR} = \begin{bmatrix} 1 & 3 & 6 \end{bmatrix}^T$,

Repeating some of the work from Example 4.2.8:

$$\bar{\mathbf{u}} \times \bar{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & 3 & 1 \\ \hat{\mathbf{j}} & 0 & 3 \\ \hat{\mathbf{k}} & 3 & 6 \end{vmatrix} = + \begin{vmatrix} 0 & 3 \\ 3 & 6 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 3 & 1 \\ 3 & 6 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 3 & 1 \\ 0 & 3 \end{vmatrix} \hat{\mathbf{k}} = -9\,\hat{\mathbf{i}} - 15\,\hat{\mathbf{j}} + 9\,\hat{\mathbf{k}}$$

$$\Rightarrow \quad A = \tfrac{1}{2} \| \bar{\mathbf{u}} \times \bar{\mathbf{v}} \| = \tfrac{1}{2} |-3| \left\| \begin{bmatrix} 3 & 5 & -3 \end{bmatrix}^T \right\| = \tfrac{3}{2} \sqrt{9 + 25 + 9} = \underline{\underline{\tfrac{3}{2} \sqrt{43}}}$$


The **scalar triple product** also has a geometric interpretation.

A parallelepiped is a six-sided object all six of whose faces are parallelograms, with pairs of opposite faces being congruent. It can be generated from a cube by repeated stretching and shearing parallel to the edges.

The volume of any parallelepiped is the product of the area of any face and the distance from that face to the opposite face:
$V = Ah$

But the area of the base is just $A = \| \bar{\mathbf{u}} \times \bar{\mathbf{v}} \|$.

The height h is the magnitude of the projection of $\bar{\mathbf{w}}$ on the normal vector to the base $\bar{\mathbf{u}} \times \bar{\mathbf{v}}$:

$$h = \left| \bar{\mathbf{w}} \bullet \left(\frac{\bar{\mathbf{u}} \times \bar{\mathbf{v}}}{\| \bar{\mathbf{u}} \times \bar{\mathbf{v}} \|} \right) \right| \quad \Rightarrow \quad V = Ah = \left\| \bar{\mathbf{u}} \times \bar{\mathbf{v}} \right\| \left| \bar{\mathbf{w}} \bullet \left(\frac{\bar{\mathbf{u}} \times \bar{\mathbf{v}}}{\| \bar{\mathbf{u}} \times \bar{\mathbf{v}} \|} \right) \right| = \left| \bar{\mathbf{w}} \bullet \bar{\mathbf{u}} \times \bar{\mathbf{v}} \right|$$

Therefore the volume of the parallelepiped is just the magnitude of the scalar triple product of the three vectors that define it.
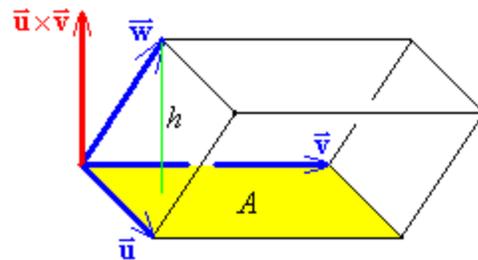
Example 4.3.2

Find the volume of the parallelepiped defined by the vectors $\bar{\mathbf{u}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, $\bar{\mathbf{v}} = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}^{\mathrm{T}}$ and $\bar{\mathbf{w}} = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}^{\mathrm{T}}$.

$$\bar{\mathbf{u}} \bullet \bar{\mathbf{v}} \times \bar{\mathbf{w}} = \begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 2 & 5 \end{vmatrix} = +1 \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} - 0 + 0 = 5 - 6 = -1$$

$$V = \left| \bar{\mathbf{u}} \bullet \bar{\mathbf{v}} \times \bar{\mathbf{w}} \right| = \left| -1 \right| = 1$$


Example 4.3.3 (Textbook, page 185, exercises 4.3, question 8)

Another method to find the distance r of a point P from the line through point P_o with line direction vector $\bar{\mathbf{d}}$:

Let θ be the angle between vectors $\bar{\mathbf{v}} = \overrightarrow{P_o P}$ and $\bar{\mathbf{d}}$.

$$\left\| \bar{\mathbf{v}} \times \bar{\mathbf{d}} \right\| = \left\| \bar{\mathbf{v}} \right\| \left\| \bar{\mathbf{d}} \right\| \sin \theta$$

But $r = \left\| \bar{\mathbf{v}} \right\| \sin \theta$

$$\Rightarrow \quad r = \frac{\left\| \overrightarrow{P_o P} \times \bar{\mathbf{d}} \right\|}{\left\| \bar{\mathbf{d}} \right\|} \; .$$
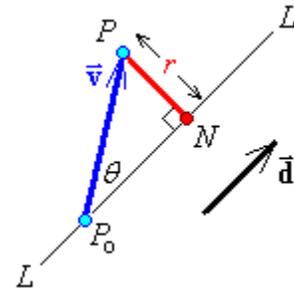
However, if one wishes to find the location of the nearest point N on the line to the point P, then the projection of $\overrightarrow{P_o P}$ on $\bar{\mathbf{d}}$ is required:

$$\overrightarrow{ON} = \overrightarrow{OP_o} + \overrightarrow{P_o N} = \overrightarrow{OP_o} + \operatorname{proj}_{\bar{\mathbf{d}}} \overrightarrow{P_o P} = \overrightarrow{OP_o} + \left(\frac{\overrightarrow{P_o P} \bullet \bar{\mathbf{d}}}{\left\| \bar{\mathbf{d}} \right\|^2} \right) \bar{\mathbf{d}}$$