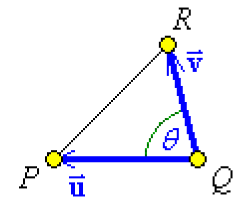
From the Final Examination of 2008 Winter, Question 11:

Given points $P(0, 3, 4)$, $Q(2, 1, 0)$ and $R(6, 3, -2)$, find:
(a) the angle *PQR*; and
(b) the area of triangle *PQR*.

(a) Let θ represent the angle *PQR*.



θ is the angle between the vectors **u** and **v**, whose tails are both at point *Q*.

$$\vec{\mathbf{u}} = \overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad \|\vec{\mathbf{u}}\| = 2\sqrt{1+1+4} = 2\sqrt{6}$$

$$\vec{\mathbf{v}} = \overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad \|\vec{\mathbf{v}}\| = 2\sqrt{4+1+1} = 2\sqrt{6}$$

and

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \left(2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}\right) \bullet \left(2\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right) = 4(-2+1-2) = 4\times(-3)$$

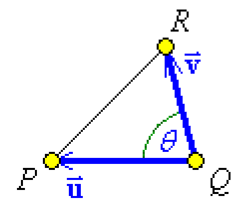
$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta \quad \Rightarrow \quad \cos\theta = \frac{\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|} = \frac{\cancel{4}(-3)}{(\cancel{2}\sqrt{6})(\cancel{2}\sqrt{6})} = -\frac{3}{6}$$

$$\cos\theta = -\frac{1}{2} \quad \Rightarrow \quad \theta = \frac{2\pi}{3} = 120°$$

Therefore angle *PQR* = 120° exactly.

[The sketch is clearly not to scale, as θ should be an obtuse angle!
The sketch serves its purpose nevertheless, for a visualization of this problem.]

(b) The area A of the triangle can be evaluated in at least two ways,
$A = \frac{1}{2}(QP)(QR)\sin\theta$ or $A = \frac{1}{2}\|\vec{\mathbf{u}}\times\vec{\mathbf{v}}\|$:



$$A = \frac{1}{2}\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\sin\theta = \frac{1}{2}\left(2\sqrt{6}\right)\left(2\sqrt{6}\right)\sin\frac{2\pi}{3} = \sqrt{6}\sqrt{6}\,2\,\frac{\sqrt{3}}{2} = 6\sqrt{3}$$

OR

$$A = \frac{1}{2}\|\vec{\mathbf{u}}\times\vec{\mathbf{v}}\|$$

$$\vec{\mathbf{u}}\times\vec{\mathbf{v}} = \left(2\begin{bmatrix}-1\\1\\2\end{bmatrix}\right)\times\left(2\begin{bmatrix}2\\1\\-1\end{bmatrix}\right) = 4\begin{vmatrix}\hat{\mathbf{i}} & -1 & 2\\ \hat{\mathbf{j}} & 1 & 1\\ \hat{\mathbf{k}} & 2 & -1\end{vmatrix}$$

$$= 4\left(\begin{vmatrix}1 & 1\\2 & -1\end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix}-1 & 2\\2 & -1\end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix}-1 & 2\\1 & 1\end{vmatrix}\hat{\mathbf{k}}\right) = 4\begin{bmatrix}-3\\3\\-3\end{bmatrix} = 12\begin{bmatrix}-1\\1\\-1\end{bmatrix}$$

$$\Rightarrow \quad A = \frac{1}{2}\|\vec{\mathbf{u}}\times\vec{\mathbf{v}}\| = \frac{1}{2}\cdot 12\sqrt{1+1+1} = 6\sqrt{3}$$

Therefore the area of the triangle PQR is $6\sqrt{3}$.