5.1 Subspaces and Spanning

Example 5.1.1

The homogeneous linear system [a homogeneous version of Assignment 2 Question 2]

$$\left\{\begin{aligned} x-2y+z&=0\\ 2x-4y+3z&=0\\ -5x+10y-2z&=0\\ 3x-6y-z&=0 \end{aligned}\right\} \quad (\text{or } AX=O)$$

has a reduced row-echelon form

$$\left[\begin{array}{ccc|c} \boxed{1} & -2 & 0 & 0\\ 0 & 0 & \boxed{1} & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{array}\right]$$

leading to the one-parameter family of solutions

$$\begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix} + t\begin{bmatrix} 2\\ 1\\ 0 \end{bmatrix}$$

Every solution is a multiple of the basic solution $\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^T$.
Geometrically this is a line through the origin in the *xy*-plane.
All solutions can be represented as a simple multiple of the basic solution.
This is an example of a one-dimensional **subspace** of $\mathbb{R}^3$.

The set consisting of the single basic vector $\left\{\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^T\right\}$ **spans** this subspace.

Example 5.1.2

A homogeneous linear system has the two-parameter family of solutions

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

All multiples of the vectors $[\ 1\ \ 2\ \ 0\ ]^T$ and $[\ 1\ \ 0\ \ 1\ ]^T$ are in this solution set.
A vector orthogonal to the solution set is

$$\vec{\mathbf{n}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{vmatrix} \hat{\mathbf{i}} & 1 & 1 \\ \hat{\mathbf{j}} & 2 & 0 \\ \hat{\mathbf{k}} & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}\hat{\mathbf{k}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

A plane through the origin with this normal vector is $-2x + y + 2z = 0$.
Therefore the solution set represents the plane $-2x + y + 2z = 0$.
This is an example of a two-dimensional subspace of $\mathbb{R}^3$.
Every solution can be represented as a linear combination of the basic solutions

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The set $\{\ [\ 1\ \ 2\ \ 0\ ]^T, [\ 1\ \ 0\ \ 1\ ]^T\ \}$ **spans** this subspace.
The solution set (subspace) U can be defined as

$$U = \text{span}\{\ [\ 1\ \ 2\ \ 0\ ]^T, [\ 1\ \ 0\ \ 1\ ]^T\ \} = \{\ s\,[\ 1\ \ 2\ \ 0\ ]^T + t\,[\ 1\ \ 0\ \ 1\ ]^T \mid s, t \in \mathbb{R}\ \}$$

A set U of vectors in $\mathbb{R}^n$ is a **subspace** of $\mathbb{R}^n$ if:

1. The zero vector **0** is in U
2. If X and Y are in U, then so is $X + Y$ (closure under addition)
3. If X is in U, then so is kX for all scalars k (closure under scalar multiplication)

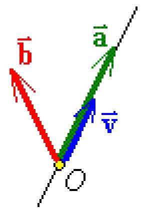
The trivial set $U = \{\ \mathbf{0}\ \}$ is a subspace of $\mathbb{R}^n$ for all n.
$\mathbb{R}^n$ is a subspace of itself.
Any subspace of $\mathbb{R}^n$ other than $\{\ \mathbf{0}\ \}$ or $\mathbb{R}^n$ is a **proper subspace** of $\mathbb{R}^n$.

For geometric vectors, lines through the origin are the only proper subspaces of $\mathbb{R}^2$, while lines and planes through the origin are the only proper subspaces of $\mathbb{R}^3$.

For non-zero non-parallel vectors **v** and **w**,
L = span { **v** } is the line through the origin parallel to vector **v** .

M = span { **v** , **w** } is the plane through the origin containing vectors **v** and **w**, with normal vector $\mathbf{n} = \mathbf{v} \times \mathbf{w}$.

Example 5.1.3



Vector **a** is in L = span { **v** } , but vector **b** is not.

$\mathbf{a} = k\mathbf{v}$ for some scalar k.
$\mathbf{b} \neq k\mathbf{v}$ for any scalar k.

If A is an $(m \times n)$ matrix, then its **null space** is the set of all solutions to the homogeneous equation $AX = O$:

$$\text{null}\, A = \{ X \text{ in } \mathbb{R}^n \mid AX = O \}$$

null A is a subspace of $\mathbb{R}^n$.

In Example 5.1.1, null A = $\text{span}\left\{ \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^T \right\}$.

The **eigenspace** of A is $E_\lambda(A) = \text{null}(\lambda I - A)$.
If $E_\lambda(A)$ contains more than the zero vector, then λ is an eigenvalue of A with corresponding eigenvectors being all members of the eigenspace except the zero vector. A spanning set for the eigenspace is a set of basic eigenvectors for that eigenvalue.

Example 5.1.4

For the subspace $U = \text{span}\{ [\,1\ \ 2\ \ 3\,]^T,\ [\,3\ \ -1\ \ 1\,]^T \}$,
determine whether the following vectors are in U:
$\mathbf{v} = [\,5\ \ 3\ \ 7\,]^T$, $\mathbf{w} = [\,4\ \ 2\ \ 6\,]^T$

If $\mathbf{v}$ is in U, then $[\,5\ \ 3\ \ 7\,]^T = s\,[\,1\ \ 2\ \ 3\,]^T + t\,[\,3\ \ -1\ \ 1\,]^T$ for some scalars s and t.

$$\Rightarrow \left\{ \begin{aligned} 1s + 3t &= 5 \\ 2s - 1t &= 3 \\ 3s + 1t &= 7 \end{aligned} \right\} \quad \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \\ 3 & 1 & 7 \end{array} \right]$$

$$\xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -7 & -7 \\ 0 & -8 & -8 \end{array} \right] \xrightarrow{R_2 \div (-7)} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & -8 & -8 \end{array} \right]$$

$$\xrightarrow[R_3 + 8R_2]{R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right] \quad \Rightarrow \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

A unique solution for s and t exists: $[\,5\ \ 3\ \ 7\,]^T = 2\,[\,1\ \ 2\ \ 3\,]^T + 1\,[\,3\ \ -1\ \ 1\,]^T$
Therefore $\vec{\mathbf{v}} \in U$.

If $\mathbf{w}$ is in U, then $[\,4\ \ 2\ \ 6\,]^T = s\,[\,1\ \ 2\ \ 3\,]^T + t\,[\,3\ \ -1\ \ 1\,]^T$ for some scalars s and t.

$$\Rightarrow \left\{ \begin{aligned} 1s + 3t &= 4 \\ 2s - 1t &= 2 \\ 3s + 1t &= 6 \end{aligned} \right\} \quad \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 3 & 1 & 6 \end{array} \right]$$

$$\xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -7 & -6 \\ 0 & -8 & -6 \end{array} \right] \xrightarrow{R_2 \div (-7)} \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & \frac{6}{7} \\ 0 & -8 & -6 \end{array} \right]$$

$$\xrightarrow[R_3 + 8R_2]{R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{10}{7} \\ 0 & 1 & \frac{6}{7} \\ 0 & 0 & \frac{6}{7} \end{array} \right] \xrightarrow[R_3 \div \frac{6}{7}]{R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{10}{7} \\ 0 & 1 & \frac{6}{7} \\ 0 & 0 & \boxed{1} \end{array} \right]$$

which is an inconsistent system. Therefore $\mathbf{w}$ is ***not*** in U.

If $U = \text{span}\{X_1, X_2, \ldots, X_k\}$ is in $\mathbb{R}^n$ and all the X_i are in a subspace W of $\mathbb{R}^n$, then $U \subseteq W$ (U is a subset of [or is the same as] W).

Example 5.1.5 (Textbook, page 201, example 5)

Given that X and Y are in $\mathbb{R}^n$, show that span { $X+Y$, $X-Y$ } = span { X, Y }.

Both $X+Y$ and $X-Y$ are clearly linear combinations of X and Y and are therefore in span { X, Y } $\Rightarrow$ span { $X+Y$, $X-Y$ } $\subseteq$ span { X, Y }.

$X = \frac{X+Y}{2} + \frac{X-Y}{2}$ and $Y = \frac{X+Y}{2} - \frac{X-Y}{2}$, so that X and Y are clearly linear combinations of $X+Y$ and $X-Y$ are therefore in span { $X+Y$, $X-Y$ }

$\Rightarrow$ span { X, Y } $\subseteq$ span { $X+Y$, $X-Y$ }.

If two sets are subsets of each other, then they are identical to each other.

Therefore span { $X+Y$, $X-Y$ } = span { X, Y }.

Example 5.1.6

Show that $\mathbf{v}$ = $[\,3\ \ 4\ \ 1\,]^T$ is in the subspace
U = span { $[\,1\ \ 1\ \ 0\,]^T$, $[\,0\ \ 1\ \ 1\,]^T$, $[\,1\ \ 2\ \ 1\,]^T$ }.

If $\mathbf{v}$ is in U, then $[\,3\ \ 4\ \ 1\,]^T = r\,[\,1\ \ 1\ \ 0\,]^T + s\,[\,0\ \ 1\ \ 1\,]^T + t\,[\,1\ \ 2\ \ 1\,]^T$
for some scalars r, s and t.

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 - R_2]{} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 3 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is a one-parameter family of solutions, so $\mathbf{v}$ is definitely in the subspace U.
One solution, setting $t = 0$, is $[\,3\ \ 4\ \ 1\,]^T = 3\,[\,1\ \ 1\ \ 0\,]^T + 1\,[\,0\ \ 1\ \ 1\,]^T$.
Another solution, setting $t = 1$, is $[\,3\ \ 4\ \ 1\,]^T = 2\,[\,1\ \ 1\ \ 0\,]^T + 1\,[\,1\ \ 2\ \ 1\,]^T$.
Therefore the three vectors { $[\,1\ \ 1\ \ 0\,]^T$, $[\,0\ \ 1\ \ 1\,]^T$, $[\,1\ \ 2\ \ 1\,]^T$ } do ***not*** provide a unique representation of vectors in the subspace U.

In fact, $[\,1\ \ 2\ \ 1\,]^T = [\,1\ \ 1\ \ 0\,]^T + [\,0\ \ 1\ \ 1\,]^T$, so that
U = span { $[\,1\ \ 1\ \ 0\,]^T$, $[\,0\ \ 1\ \ 1\,]^T$, $[\,1\ \ 2\ \ 1\,]^T$ } = span { $[\,1\ \ 1\ \ 0\,]^T$, $[\,0\ \ 1\ \ 1\,]^T$ }.

span { $[\,1\ \ 1\ \ 0\,]^T$, $[\,0\ \ 1\ \ 1\,]^T$ } is a “better” spanning set for the subspace U.
All members of U can be written as a linear combination of $[\,1\ \ 1\ \ 0\,]^T$ and $[\,0\ \ 1\ \ 1\,]^T$ in only one way.

A one-dimensional subspace (a line) needs only one non-zero vector in its spanning set.
A two-dimensional subspace (a plane) needs only two non-zero non-parallel vectors in its spanning set.
$\mathbb{R}^3$ needs only three non-zero vectors (not all in the same plane) in its spanning set.

The **standard basis** for $\mathbb{R}^3$ is $\left\{ \hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

These are also the columns E_1, E_2, E_3 of the identity matrix I_3.

Therefore $\mathbb{R}^3 = \text{span}\{ E_1, E_2, E_3 \} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

The standard basis for $\mathbb{R}^2$ is $\{ E_1, E_2 \} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

5.2 Independence and Dimension

As in Example 5.1.6 above, suppose that two linear combinations of vectors in $\mathbb{R}^n$ both represent the same vector *Y*:

$$Y = r_1 X_1 + r_2 X_2 + \ldots + r_k X_k = s_1 X_1 + s_2 X_2 + \ldots + s_k X_k$$

$$\Rightarrow \quad (r_1 - s_1) X_1 + (r_2 - s_2) X_2 + \ldots + (r_k - s_k) X_k = 0$$

If there is only one possible representation of Y as a linear combination of the $\{ X_i \}$, then we must have $s_i = r_i$ for all *i*, so that all of the coefficients $(r_i - s_i)$ above must be zero. When the linear combination is unique, the set $\{ X_i \}$ is **independent**.

If a set of vectors is independent, then it is impossible to write any of them as a linear combination of the others. This leads to the related test for independence:

The set of vectors $\{ X_1, X_2, \ldots, X_k \}$ is **linearly independent** if and only if

> the only solution to the equation $c_1 X_1 + c_2 X_2 + \ldots + c_k X_k = 0$
> is the trivial solution $c_1 = c_2 = \ldots = c_k = 0$.

Example 5.2.1

Show that the set { [1 1 0]^T , [0 1 1]^T , [1 2 1]^T } (from Example 5.1.6) is not independent.

Solve, for *r*, *s*, *t*, the equation $r\,[\,1\;1\;0\,]^T + s\,[\,0\;1\;1\,]^T + t\,[\,1\;2\;1\,]^T = [\,0\;0\;0\,]^T$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - R_2]{} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Again this is a one-parameter family of solutions.
Non-trivial solutions for *r*, *s*, *t* exist ($r = s = -t$, where *t* is free to be any real number).
Therefore the set is not independent.
In fact $-1\,[\,1\;1\;0\,]^T - 1\,[\,0\;1\;1\,]^T + 1\,[\,1\;2\;1\,]^T = [\,0\;0\;0\,]^T$
$\Rightarrow \; [\,1\;2\;1\,]^T = [\,1\;1\;0\,]^T + [\,0\;1\;1\,]^T$
which clearly establishes that one of the vectors is dependent on the other two.

The standard basis { E_1, E_2 } for $\mathbb{R}^2$ is independent.

The standard basis { E_1, E_2, E_3 } for $\mathbb{R}^3$ is independent.

The zero vector is not a member of any set of independent vectors.
[Reason: $kO + r_1X_1 + r_2X_2 + \ldots + r_nX_n = O$ clearly has non-trivial solutions
$r_1 = r_2 = ... = r_n = 0$, k = any real number.]

A set containing a single vector { X } is independent if and only if $X \neq \mathbf{0}$.

Any **basis** of a subspace is both independent and spans the subspace.
The number of vectors in a basis is the **dimension** of the subspace (dim(U)).

Any independent set in a subspace U can be enlarged (by adding vectors) to a basis for U.
Any set that spans a subspace U can be reduced (by deleting vectors) to a basis for U.

In Example 5.2.1, deletion of $[\ 1\ \ 2\ \ 1\ ]^T$ leaves a basis { $[\ 1\ \ 1\ \ 0\ ]^T$, $[\ 0\ \ 1\ \ 1\ ]^T$ } for U.

If the number of vectors m in a set A equals the dimension of the subspace U, then the following are either all true or all false:

- A is independent;
- A spans U;
- A is a basis for U.

Therefore, when m = dim(U), it is sufficient to test ***one*** of independence or spanning to determine whether or not A is a basis for U.

Example 5.2.2

Suppose that { X, Y, Z } is a basis for $\mathbb{R}^3$.
Show that { $X + aZ$, Y, Z } is also a basis for $\mathbb{R}^3$ for any choice of the scalar a.

Test for independence:
$r(X + aZ) + sY + tZ = rX + sY + (t+a)Z = 0$
But { X, Y, Z } is a basis for $\mathbb{R}^3$ $\Rightarrow$ the only solution to $rX + sY + uZ = 0$ is the trivial solution $r = s = u = 0$.
Let $u = t+a$, then the only solution to $r(X + aZ) + sY + tZ = 0$ is $r = s = 0$, $t = -a$.
Therefore { $X + aZ$, Y, Z } are independent.
dim($\mathbb{R}^3$) = 3 = number of vectors in the set.
Therefore { $X + aZ$, Y, Z } is also a basis for $\mathbb{R}^3$ for any choice of the scalar a.

Example 5.2.3

Determine if this set of vectors is a basis of $\mathbb{R}^3$:
$A = \{ [\,1\ \ 0\ \ 2\,]^T,\ [\,1\ \ -2\ \ 3\,]^T,\ [\,1\ \ 4\ \ 0\,]^T \}$

$n(A) = \dim(\mathbb{R}^3) = 3$
Testing for independence:
EITHER
$r\,[\,1\ \ 0\ \ 2\,]^T + s\,[\,1\ \ -2\ \ 3\,]^T + t\,[\,1\ \ 4\ \ 0\,]^T = [\,0\ \ 0\ \ 0\,]^T$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 4 & 0 \\ 2 & 3 & 0 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & 4 & 0 \end{array}\right] \xrightarrow[R_3 + 2R_2]{R_1 - R_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 3 & 0 \\ 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Non-trivial solutions therefore exist for r, s, t.
$t\,(-3\,[\,1\ \ 0\ \ 2\,]^T + 2\,[\,1\ \ -2\ \ 3\,]^T + 1\,[\,1\ \ 4\ \ 0\,]^T) = [\,0\ \ 0\ \ 0\,]^T$ for all real numbers t
$\Rightarrow\ [\,1\ \ 4\ \ 0\,]^T = 3\,[\,1\ \ 0\ \ 2\,]^T - 2\,[\,1\ \ -2\ \ 3\,]^T$
Therefore the set A is not independent.

OR

One can spot that $[\,1\ \ 4\ \ 0\,]^T = 3\,[\,1\ \ 0\ \ 2\,]^T - 2\,[\,1\ \ -2\ \ 3\,]^T$, so that one vector is a linear combination of the others. Immediately we can conclude that set A is not independent.

Therefore set A is not a basis (and does not span $\mathbb{R}^3$ either).

The reduced set $\{ [\,1\ \ 0\ \ 2\,]^T, [\,1\ \ -2\ \ 3\,]^T \}$ is independent, but does not span $\mathbb{R}^3$.

Adding the vector $E_1 = [\,1\ \ 0\ \ 0\,]^T$ to the reduced set does generate a basis for $\mathbb{R}^3$:
$B = \{ [\,1\ \ 0\ \ 0\,]^T, [\,1\ \ 0\ \ 2\,]^T, [\,1\ \ -2\ \ 3\,]^T \}$

Example 5.2.4

The subspace U of $\mathbb{R}^3$ is defined by

$$U = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 11 \\ 0 \\ 2 \end{bmatrix} \right\}$$

(a) Is $[\,3\ \ 1\ \ 1\,]^T$ in U?
(b) Is $[\,3\ \ 1\ \ 0\,]^T$ in U?
(c) Find a basis for U.
(d) Write down $\dim U$.
(e) Is $U = \mathbb{R}^3$?

(a) Solve $[\,3\ \ 1\ \ 1\,]^T = r\,[\,1\ \ 4\ \ 2\,]^T + s\,[\,2\ \ -3\ \ -1\,]^T + t\,[\,11\ \ 0\ \ 2\,]^T$ for r, s, t:

$$\left[\begin{array}{ccc|c} 1 & 2 & 11 & 3 \\ 4 & -3 & 0 & 1 \\ 2 & -1 & 2 & 1 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & 2 & 11 & 3 \\ 0 & -11 & -44 & -11 \\ 0 & -5 & -20 & -5 \end{array}\right]$$

$$\xrightarrow{R_2 \div (-11)} \left[\begin{array}{ccc|c} 1 & 2 & 11 & 3 \\ 0 & 1 & 4 & 1 \\ 0 & -5 & -20 & -5 \end{array}\right] \xrightarrow[R_3 + 5R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is a one-parameter family of solutions,
$r = 1 - 3t,\ \ s = 1 - 4t,\ \ t \in \mathbb{R}$
One solution (setting $t = 0$) is $[\,3\ \ 1\ \ 1\,]^T = [\,1\ \ 4\ \ 2\,]^T + [\,2\ \ -3\ \ -1\,]^T$.
Another solution (setting $t = 1$) is
$[\,3\ \ 1\ \ 1\,]^T = -2\,[\,1\ \ 4\ \ 2\,]^T + -3\,[\,2\ \ -3\ \ -1\,]^T + [\,11\ \ 0\ \ 2\,]^T$
The representation of $[\,3\ \ 1\ \ 1\,]^T$ is not unique.

Therefore $\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 11 \\ 0 \\ 2 \end{bmatrix} \right\}$ is not a basis for U and the three vectors are not independent ($[\,11\ \ 0\ \ 2\,]^T = 3\,[\,1\ \ 4\ \ 2\,]^T + 4\,[\,2\ \ -3\ \ -1\,]^T$).
However, $[\,3\ \ 1\ \ 1\,]^T$ is in U.

Example 5.2.4 (continued)

(b) Solve $[\,3\ \ 1\ \ 0\,]^T = r\,[\,1\ \ 4\ \ 2\,]^T + s\,[\,2\ \ -3\ \ -1\,]^T + t\,[\,11\ \ 0\ \ 2\,]^T$ for r, s, t:

$$\left[\begin{array}{ccc|c} 1 & 2 & 11 & 3 \\ 4 & -3 & 0 & 1 \\ 2 & -1 & 2 & 0 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & 2 & 11 & 3 \\ 0 & -11 & -44 & -11 \\ 0 & -5 & -20 & -6 \end{array}\right]$$

$$\xrightarrow{R_2 \div (-11)} \left[\begin{array}{ccc|c} 1 & 2 & 11 & 3 \\ 0 & 1 & 4 & 1 \\ 0 & -5 & -20 & -6 \end{array}\right] \xrightarrow[R_3 + 5R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

$$\xrightarrow[R_3 \times (-1)]{} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

which is an inconsistent system.
Therefore $[\,3\ \ 1\ \ 0\,]^T$ is not in U.

(c) Vectors $[\,1\ \ 4\ \ 2\,]^T$ and $[\,2\ \ -3\ \ -1\,]^T$ are clearly independent (neither is a multiple of the other). A basis for a subspace is a set of independent vectors that span that subspace. $U = \text{span}\,\{\,[\,1\ \ 4\ \ 2\,]^T,\ [\,2\ \ -3\ \ -1\,]^T\,\}$. Therefore a basis for U is $\{\,[\,1\ \ 4\ \ 2\,]^T,\ [\,2\ \ -3\ \ -1\,]^T\,\}$.

In fact, any two of the original three vectors will serve as a basis for U.

(d) The basis for U contains two vectors. Therefore $\dim U = 2$.

(e) $\dim U = 2$ but $\dim \mathbb{R}^3 = 3$. Therefore $U \neq \mathbb{R}^3$.
In fact $U \subset \mathbb{R}^3$ (geometrically U is a plane through the origin).

END OF CHAPTER 5
END OF MATH 2050 !