MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

FINAL EXAMINATION **MATHEMATICS 2050** WINTER 2009

Solutions by Dr. George (Instructor for Section 4)

1. (a) Use Gaussian elimination to solve the following system of linear equations [6]

$$\begin{aligned} 2x - y \quad\quad + w &= 2 \\ x \quad - 3z + 4w &= -1 \\ -x + y - 3z + 3w &= -3 \end{aligned}$$

$$\left[\begin{array}{rrrr|r} 2 & -1 & 0 & 1 & 2 \\ 1 & 0 & -3 & 4 & -1 \\ -1 & 1 & -3 & 3 & -3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{rrrr|r} 1 & 0 & -3 & 4 & -1 \\ 2 & -1 & 0 & 1 & 2 \\ -1 & 1 & -3 & 3 & -3 \end{array}\right]$$

$$\xrightarrow[R_3 + R_1]{R_2 - 2R_1} \left[\begin{array}{rrrr|r} 1 & 0 & -3 & 4 & -1 \\ 0 & -1 & 6 & -7 & 4 \\ 0 & 1 & -6 & 7 & -4 \end{array}\right] \xrightarrow[R_3 + R_2]{} \left[\begin{array}{rrrr|r} 1 & 0 & -3 & 4 & -1 \\ 0 & -1 & 6 & -7 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \times (-1)} \left[\begin{array}{rrrr|r} \boxed{1} & 0 & -3 & 4 & -1 \\ 0 & \boxed{1} & -6 & 7 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left\{ \begin{array}{c} x - 3z + 4w = -1 \\ y - 6z + 7w = -4 \end{array} \right\}, \text{ with } z, w = \text{ free parameters.}$$

$\Rightarrow$ a two-parameter family of solutions:

$$\left\{ \begin{array}{c} x = 3s - 4t - 1 \\ y = 6s - 7t - 4 \\ z = s \\ w = t \end{array} \right\}, \qquad (s \in \mathbb{R},\ t \in \mathbb{R})$$

1(a) (continued)

or

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s\begin{bmatrix} 3 \\ 6 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ -7 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -4 \\ 0 \\ 0 \end{bmatrix}, \qquad (s \in \mathbb{R},\ t \in \mathbb{R})$$

Note: if w and x are chosen as the leading variables and y and z as the free parameters, then the solution takes the acceptable equivalent form

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s\begin{bmatrix} -\frac{3}{7} \\ 0 \\ 1 \\ \frac{6}{7} \end{bmatrix} + t\begin{bmatrix} \frac{4}{7} \\ 1 \\ 0 \\ -\frac{1}{7} \end{bmatrix} + \begin{bmatrix} \frac{9}{7} \\ 0 \\ 0 \\ -\frac{4}{7} \end{bmatrix}, \qquad (s \in \mathbb{R},\ t \in \mathbb{R})$$

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1 (b) For which values of k does the system [4]

$$\begin{aligned} 2x - y + 3z &= -2 \\ 7y + 2z &= 4 \\ (9-k^2)z &= 27-k^3 \end{aligned}$$

have (i) a unique solution? (ii) no solution? (iii) infinitely many solutions? (Do not find the solutions.)

This system is in triangular form.
It is obvious that this system will lead to a unique solution if and only if $9-k^2 \neq 0$.
$9-k^2 = 0 \quad \Rightarrow \quad k = \pm 3$
If $k = 3$ then row 3 becomes $[\ 0\ \ 0\ \ 0\ \mid\ 0\ ]$ and the system has a one-parameter family of solutions (infinitely many solutions).

If $k = -3$ then row 3 becomes $[\ 0\ \ 0\ \ 0\ \mid\ 54\ ]$ and the system is inconsistent (no solution).

Therefore

(i) a unique solution for all $k \neq \pm 3$
(ii) no solution for $k = -3$
(iii) infinitely many solutions for $k = 3$

2. Given the matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -4 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix},$$

find, if possible:
(a) AB (b) AC (c) a matrix X such that $DXC^{-1} = D^{-1}$. [2, 1, 4]

(a) $$AB = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -4 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -2 & 0 \\ -9 & 2 \end{bmatrix}}}$$

(b) AC **<u>does not exist</u>**
[the following reason need not be given to gain the mark:
Examining dimensions: ((2×**3**) × (**2**×2)) has a mismatch in the internal dimensions.]

(c) $$DXC^{-1} = D^{-1} \quad \Rightarrow \quad D^{-1}DXC^{-1}C = D^{-1}D^{-1}C \quad \Rightarrow \quad X = \left(D^{-1}\right)^2 C$$

$$D^{-1} = \frac{1}{-2+1}\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow \left(D^{-1}\right)^2 = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\Rightarrow X = \left(D^{-1}\right)^2 C = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -1 & 0 \\ -4 & -2 \end{bmatrix}}}$$

OR

$$\Rightarrow D^{-1}C = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -3 & -2 \end{bmatrix}$$

$$\Rightarrow X = D^{-1}\left(D^{-1}C\right) = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -2 & -2 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -4 & -2 \end{bmatrix}$$

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3. Find matrix A, given that $2A^{\mathrm{T}} + \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix} = \left(\begin{bmatrix} 5 & 1 \\ -1 & 6 \end{bmatrix} - A \right)^{\mathrm{T}}$. [5]

$$2A^{\mathrm{T}} + \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix} = \left(\begin{bmatrix} 5 & 1 \\ -1 & 6 \end{bmatrix} - A \right)^{\mathrm{T}} \Rightarrow 2A + \begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -1 & 6 \end{bmatrix} - A$$

$$\Rightarrow 2A + A = \begin{bmatrix} 5 & 1 \\ -1 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix} \Rightarrow A = \underline{\underline{\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}}}$$

OR

$$2A^{\mathrm{T}} + \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix} = \left(\begin{bmatrix} 5 & 1 \\ -1 & 6 \end{bmatrix} - A \right)^{\mathrm{T}} = \begin{bmatrix} 5 & -1 \\ 1 & 6 \end{bmatrix} - A^{\mathrm{T}}$$

$$\Rightarrow 2A^{\mathrm{T}} + A^{\mathrm{T}} = \begin{bmatrix} 5 & -1 \\ 1 & 6 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix} \Rightarrow 3A^{\mathrm{T}} = \begin{bmatrix} 3 & 0 \\ -3 & 6 \end{bmatrix}$$

$$\Rightarrow 3A = \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

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4. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -1 \\ 3 & 1 & 1 \end{bmatrix}$.

(a) Find A^{-1}. [5]

The matrix of cofactors is

$$C = \begin{bmatrix} +\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & -4 & 7 \\ 0 & -2 & 2 \\ 1 & 2 & -3 \end{bmatrix}$$

4(a) (continued)

$$\Rightarrow \quad \operatorname{adj} A = C^{\mathrm{T}} = \begin{bmatrix} -1 & 0 & 1 \\ -4 & -2 & 2 \\ 7 & 2 & -3 \end{bmatrix}$$

Expanding along row 1:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 1\times(-1) + 1\times(-4) + 1\times 7 = 2$$

$$A^{-1} = \frac{\operatorname{adj} A}{\det A} = \underline{\underline{\frac{1}{2}\begin{bmatrix} -1 & 0 & 1 \\ -4 & -2 & 2 \\ 7 & 2 & -3 \end{bmatrix}}} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ -2 & -1 & 1 \\ \frac{7}{2} & 1 & -\frac{3}{2} \end{bmatrix}$$

OR

Use Gaussian elimination to find the inverse:

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 - R_1]{R_3 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -2 & -1 & 1 & 0 \\ 0 & -2 & -2 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \div (-3)} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -2 & -2 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 + 2R_2]{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} & -\frac{7}{3} & -\frac{2}{3} & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \div \left(-\frac{2}{3}\right)]{} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{7}{2} & 1 & -\frac{3}{2} \end{array}\right]$$

$$\xrightarrow[R_2 - \frac{2}{3}R_3]{R_1 - \frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -2 & -1 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & 1 & -\frac{3}{2} \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ -2 & -1 & 1 \\ \frac{7}{2} & 1 & -\frac{3}{2} \end{bmatrix}$$

Valid alternative sequences of row operations exist.

4 (b) Use A^{-1} to solve the system [2]

$$\begin{aligned} x + y + z &= 1 \\ x - 2y - z &= -1 \\ 3x + y + z &= 3 \end{aligned}$$

$$AX = B \quad \Rightarrow \quad X = A^{-1}B$$

$$\Rightarrow \quad X = \frac{1}{2}\begin{bmatrix} -1 & 0 & 1 \\ 4 & -2 & 2 \\ 7 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Therefore $x = 1$, $y = 2$, $z = -2$.

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4 (c) Let $C = [\,-3 \quad 1 \quad 5\,]^{T}$. Use Cramer's rule to find the value of y in the solution of the system $AX = C$. [4]

$$AX = C \quad \Rightarrow \quad y = \frac{\det A_2}{\det A} = \frac{\begin{vmatrix} 1 & -3 & 1 \\ 1 & 1 & -1 \\ 3 & 5 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -1 \\ 3 & 1 & 1 \end{vmatrix}} = \frac{1\begin{vmatrix} 1 & -1 \\ 5 & 1 \end{vmatrix} - 1\begin{vmatrix} -3 & 1 \\ 5 & 1 \end{vmatrix} + 3\begin{vmatrix} -3 & 1 \\ 1 & -1 \end{vmatrix}}{2}$$

$$= \frac{6+8+6}{2} = \frac{20}{2} \quad \Rightarrow \quad y = \underline{\underline{10}}$$

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5. (a) Use row operations to show that [5]

$$\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_1-b_1 & a_2-b_2 & a_3-b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -2\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$LHS = \begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_1-b_1 & a_2-b_2 & a_3-b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \xrightarrow{R_1+R_2} \begin{vmatrix} 2a_1 & 2a_2 & 2a_3 \\ a_1-b_1 & a_2-b_2 & a_3-b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\xrightarrow{R_2-\frac{1}{2}R_1} \begin{vmatrix} 2a_1 & 2a_2 & 2a_3 \\ -b_1 & -b_2 & -b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Extracting row factors:

$$LHS = 2\times(-1)\times\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = RHS$$

Valid alternative sequences of row operations exist, but the above is the most concise.

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5 (b) Find $\det D$, given that D is a 4×4 matrix and $\det(2D) = -32$. [3]

$\det(kA) = k^n \det A$ for any $(n\times n)$ matrix A. Here $n = 4$

$\Rightarrow \det(2D) = 2^4 \det D \Rightarrow -32 = 16\det D \Rightarrow \det D = \underline{\underline{-2}}$

6. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

(a) Find the eigenvalues and corresponding eigenvectors of A. [8]

The eigenvalues λ are the solutions to the characteristic equation $\det(\lambda I - A) = 0$.

$$\begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = 0 \quad \Rightarrow \quad (\lambda-2)^2 - 1 = 0 \quad \Rightarrow \quad (\lambda-2)^2 = 1$$

$$\Rightarrow \quad \lambda - 2 = \pm 1 \quad \Rightarrow \quad \underline{\underline{\lambda = 1}} \text{ or } \underline{\underline{\lambda = 3}}$$ [**2 marks**]

(**or** solve the quadratic equation $\lambda^2 - 4\lambda + 4 - 1 = 0$).

For each λ, the eigenvectors are the non-trivial solutions to $(\lambda I - A)X = O$.

For $\lambda = 1$:

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad -x - y = 0 \quad \Rightarrow \quad y = -x$$

$\Rightarrow$ the 1-eigenvectors are $\underline{\underline{t\begin{bmatrix} 1 \\ -1 \end{bmatrix}}}, \; (t \neq 0)$ $\quad \left[\text{or } t\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \; (t \neq 0)\right]$ [**3 marks**]

For $\lambda = 3$:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad x - y = 0 \quad \Rightarrow \quad y = x$$

$\Rightarrow$ the 3-eigenvectors are $\underline{\underline{t\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}, \; (t \neq 0)$ [**3 marks**]

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6 (b) Diagonalize A. That is, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. [3]

Immediately from 6(a),

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Valid alternatives include

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}; \qquad P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}; \quad \text{etc.}$$

(c) Use the information from part (b) to calculate A^6. [3]

$$P^{-1}AP = D \quad \Rightarrow \quad A = PDP^{-1} \quad \Rightarrow \quad A^6 = \left(PDP^{-1}\right)^6 = \ldots = PD^6P^{-1}$$

$$P^{-1} = \frac{1}{1+1}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

The alternatives in (b) produce different forms for P^{-1} here.

For the main answer in 6(b) above,

$$A^6 = PD^6P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 729 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 729 & 729 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 730 & 728 \\ 728 & 730 \end{bmatrix} \quad \Rightarrow \quad A^6 = \begin{bmatrix} 365 & 364 \\ 364 & 365 \end{bmatrix}$$

OR

$$A^6 = PD^6P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 729 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 729 \\ -1 & 729 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 730 & 728 \\ 728 & 730 \end{bmatrix} \quad \Rightarrow \quad A^6 = \begin{bmatrix} 365 & 364 \\ 364 & 365 \end{bmatrix}$$

7. Let $\vec{\mathbf{u}} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$ and $\vec{\mathbf{v}} = \begin{bmatrix} 3 & 5 & -2 \end{bmatrix}^T$. [5]
Find a vector of length 2 that is perpendicular to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

A vector that is perpendicular to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$:

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & 1 & 3 \\ \hat{\mathbf{j}} & 2 & 5 \\ \hat{\mathbf{k}} & 1 & -2 \end{vmatrix} = \begin{bmatrix} -9 \\ 5 \\ -1 \end{bmatrix}$$

If the vectors are taken in the opposite order, then the correct vector is $\vec{\mathbf{v}} \times \vec{\mathbf{u}} = \begin{bmatrix} 9 \\ -5 \\ 1 \end{bmatrix}$.

$$\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \sqrt{(-9)^2 + 5^2 + (-1)^2} = \sqrt{81+25+1} = \sqrt{107}$$

A unit vector in the desired direction is $\frac{1}{\sqrt{107}} \begin{bmatrix} -9 \\ 5 \\ -1 \end{bmatrix} \left(\text{or } \frac{1}{\sqrt{107}} \begin{bmatrix} 9 \\ -5 \\ 1 \end{bmatrix} \right)$

The required vector of length 2 is $\underline{\underline{\frac{2}{\sqrt{107}} \begin{bmatrix} -9 \\ 5 \\ -1 \end{bmatrix}}} \left(\text{or } \frac{2}{\sqrt{107}} \begin{bmatrix} 9 \\ -5 \\ 1 \end{bmatrix} \right)$

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8. Points $A(1, 1, 4)$, $B(2,-3, 5)$, and $C(4,-8, 10)$ are given.
(a) Find angle B of triangle ABC. [4]

$$\overrightarrow{BA} = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix} \text{ and } \overrightarrow{BC} = \begin{bmatrix} 2 \\ -5 \\ 5 \end{bmatrix} \quad \Rightarrow \quad \cos B = \frac{\overrightarrow{BA} \bullet \overrightarrow{BC}}{\|\overrightarrow{BA}\| \|\overrightarrow{BC}\|} =$$

$$\frac{\begin{bmatrix} -1 & 4 & -1 \end{bmatrix}^T \bullet \begin{bmatrix} 2 & -5 & 5 \end{bmatrix}^T}{\sqrt{1+16+1} \cdot \sqrt{4+25+25}} = \frac{-2-20-5}{\sqrt{18}\sqrt{54}} = \frac{-27}{\sqrt{2\times9\times3\times2\times9}} = \frac{-27}{2\times9\sqrt{3}} = -\frac{\sqrt{3}}{2}$$

$$\cos B = -\frac{\sqrt{3}}{2} \quad \Rightarrow \quad \underline{\underline{B = \frac{5\pi}{6}}} \quad (= 150°)$$

8(b) Find the area of triangle ABC. [4]

$$\text{Area} = \tfrac{1}{2} BA \cdot BC \cdot \sin B = \frac{1}{2}\sqrt{18}\sqrt{54}\sin\left(\frac{5\pi}{6}\right) = \frac{1}{\not{2}}\left(3\not{\sqrt{2}}\right)\left(3\sqrt{3}\not{\sqrt{2}}\right)\frac{1}{2} = \underline{\underline{\frac{9\sqrt{3}}{2}}}$$

OR

$$\overrightarrow{BA}\times\overrightarrow{BC} = \begin{vmatrix} \hat{\mathbf{i}} & -1 & 2 \\ \hat{\mathbf{j}} & 4 & -5 \\ \hat{\mathbf{k}} & -1 & 5 \end{vmatrix} = \begin{bmatrix} 15 \\ 3 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad \text{Area} = \frac{1}{2}\left\| \overrightarrow{BA}\times\overrightarrow{BC} \right\| = \frac{1}{2}3\sqrt{25+1+1} = \frac{9\sqrt{3}}{2}$$

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9 (a) Find the point of intersection of the two lines $x = -1 - 3t,\ y = 2 + 2t,\ z = 3 - t$ and $x = 2 + 5s,\ y = -s,\ z = 4 + 3s$. [5]

At any point of intersection,

$$\left\{\begin{aligned} x &= -1-3t = 2+5s \\ y &= 2+2t = 0-s \\ z &= 3-t = 4+3s \end{aligned}\right\}$$

$$\Rightarrow \left\{\begin{aligned} y:&\quad 1s+2t = -2 \\ x:&\quad 5s+3t = -3 \\ z:&\quad 3s+1t = -1 \end{aligned}\right\} \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 5 & 3 & -3 \\ 3 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[R_3-3R_1]{R_2-5R_1} \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -7 & 7 \\ 0 & -5 & 5 \end{array}\right] \xrightarrow{R_2\div(-7)} \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & -5 & 5 \end{array}\right]$$

$$\xrightarrow[R_3+5R_2]{R_1-2R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad s = 0,\ t = -1$$

(unique solution)

$$\Rightarrow \quad x = 2+0, \quad y = 0, \quad z = 4+0$$

Therefore the lines meet at the point $\underline{\underline{(2, 0, 4)}}$.

9 (b) Find an equation of the plane containing the two lines of part (a). [4]

The line direction vectors are $\vec{\mathbf{d}}_1 = \begin{bmatrix} -3 & 2 & -1 \end{bmatrix}^T$ and $\vec{\mathbf{d}}_2 = \begin{bmatrix} 5 & -1 & 3 \end{bmatrix}^T$

A normal vector to the plane containing these two lines is:

$$\vec{\mathbf{d}}_1 \times \vec{\mathbf{d}}_2 = \begin{vmatrix} \hat{\mathbf{i}} & -3 & 5 \\ \hat{\mathbf{j}} & 2 & -1 \\ \hat{\mathbf{k}} & -1 & 3 \end{vmatrix} = \begin{bmatrix} 5 \\ 4 \\ -7 \end{bmatrix} = \vec{\mathbf{n}}$$

The displacement vector to a point on the plane is $\vec{\mathbf{a}} = \begin{bmatrix} 2 & 0 & 4 \end{bmatrix}^T$

$$\vec{\mathbf{n}} \bullet \vec{\mathbf{a}} = \begin{bmatrix} 5 \\ 4 \\ -7 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = 10 + 0 - 28 = -18$$

The equation of the plane is $\vec{\mathbf{n}} \bullet \vec{\mathbf{p}} = \vec{\mathbf{n}} \bullet \vec{\mathbf{a}}$ or

$$\underline{\underline{5x + 4y - 7z = -18}}$$

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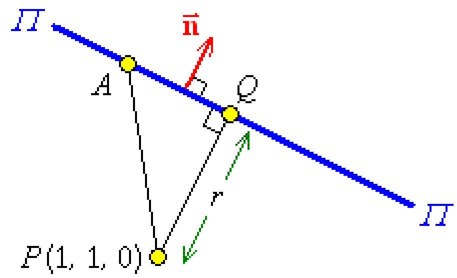
10. Find the distance from the point $P(1, 1, 0)$ to the plane $x + y - z = 1$, and find the point Q on the plane which is closest to P. [8]

A normal vector to the plane $x + y - z = 1$ is $\vec{\mathbf{n}} = [\,1 \quad 1 \quad -1\,]^T$.

A point A on the plane is $(1, 0, 0)$



$\Rightarrow \quad \overrightarrow{PA} = [\,0 \quad -1 \quad 0\,]^T$

$$\overrightarrow{PQ} = \text{proj}_{\vec{\mathbf{n}}}\overrightarrow{PA} = \left(\overrightarrow{PA}\bullet\hat{\mathbf{n}}\right)\hat{\mathbf{n}} = \left(\frac{\overrightarrow{PA}\bullet\vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|^2}\right)\vec{\mathbf{n}}$$

$$= \frac{[\,0 \quad -1 \quad 0\,]^T\bullet[\,1 \quad 1 \quad -1\,]^T}{1^2 + 1^2 + (-1)^2}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \frac{-1}{3}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ} = \frac{1}{3}\left(3\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}-1\\-1\\1\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}2\\2\\1\end{bmatrix}$$

Therefore the point Q is at $\underline{\underline{\left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}\right)}}$

and

$$r = \left\|\overrightarrow{PQ}\right\| = \frac{1}{3}\sqrt{(-1)^2 + (-1)^2 + 1^2} = \underline{\underline{\frac{\sqrt{3}}{3}}}$$

Some valid variations are possible.

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11. Let $Y = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}^T$ and $Z = \begin{bmatrix} -1 & -2 & -1 \end{bmatrix}^T$ be two vectors in $\mathbb{R}^3$.

(a) Show that $\{ Y, Z \}$ is linearly independent. [4]

Clearly $Y \neq kZ$ for any scalar k.
Therefore $\{ Y, Z \}$ is linearly independent.

OR

Seek non-trivial solutions (s, t) to the homogeneous system $sY + tZ = O$:

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{array}\right] \xrightarrow[R_3 - 3R_2]{} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

which clearly has a unique solution $s = t = 0$ – the trivial solution only.
Therefore $\{ Y, Z \}$ is linearly independent.

11(b) Determine whether $X = \begin{bmatrix} 6 & 5 & -1 \end{bmatrix}^T$ lies in $U = \text{span}\{ Y, Z \}$. [4]

Solve $X = sY + tZ$ for s and t:

$$\left[\begin{array}{cc|c} 2 & -1 & 6 \\ 1 & -2 & 5 \\ 1 & -1 & -1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 2 & -1 & 6 \\ 1 & -1 & -1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 3 & -4 \\ 0 & 1 & -6 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -6 \\ 0 & 3 & -4 \end{array}\right] \xrightarrow[R_3 - 3R_2]{} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -6 \\ 0 & 0 & 14 \end{array}\right]$$

which is clearly inconsistent (row 3 is $0s + 0t = 14$).

Therefore **NO**, $X = \begin{bmatrix} 6 & 5 & -1 \end{bmatrix}^T$ does not lie in $U = \text{span}\{ Y, Z \}$.

11(b) (continued)
OR

In $\mathbb{R}^3$, X lies in span $\{ Y, Z \}$ if and only if $X \bullet Y \times Z = 0$

$$X \bullet Y \times Z = \begin{vmatrix} 6 & 2 & -1 \\ 5 & 1 & -2 \\ -1 & 1 & -1 \end{vmatrix} = (-1)\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 6 & -1 \\ 5 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 6 & 2 \\ 5 & 1 \end{vmatrix}$$

$$= -\big((-4+1)+(-12+5)+(6-10)\big) = -(-3-7-4) = +14 \neq 0$$

Therefore <u>**NO**</u>, $X = [\,6 \quad 5 \quad -1\,]^{\mathrm{T}}$ does not lie in $U = \text{span}\{ Y, Z \}$.

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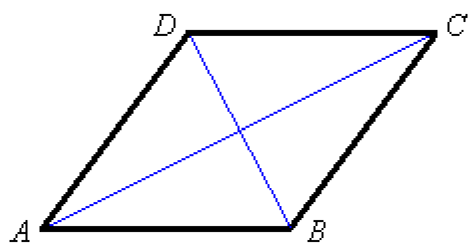
12. Do <u>ONE</u> of the following: [7]

(a) A parallelogram with sides of equal length is called a rhombus.
Use vectors to prove that the diagonals of a rhombus are perpendicular.



Let $\vec{\mathbf{u}} = \overrightarrow{AB} = \overrightarrow{DC}$ and $\vec{\mathbf{v}} = \overrightarrow{BC} = \overrightarrow{AD}$

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{\mathbf{u}} + \vec{\mathbf{v}}$$

$$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -\vec{\mathbf{u}} + \vec{\mathbf{v}}$$

$$\Rightarrow \quad \overrightarrow{AC} \bullet \overrightarrow{BD} = (\vec{\mathbf{v}} + \vec{\mathbf{u}}) \bullet (\vec{\mathbf{v}} - \vec{\mathbf{u}})$$

$$= \vec{\mathbf{v}} \bullet \vec{\mathbf{v}} + \cancel{\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}} - \cancel{\vec{\mathbf{v}} \bullet \vec{\mathbf{u}}} - \vec{\mathbf{u}} \bullet \vec{\mathbf{u}} = \|\vec{\mathbf{v}}\|^2 - \|\vec{\mathbf{u}}\|^2$$

But $ABCD$ is a rhombus $\Rightarrow$ $AD = AB$ $\Rightarrow$ $\|\vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\|$

Therefore $\overrightarrow{AC} \bullet \overrightarrow{BD} = 0$ $\Rightarrow$ the diagonals are perpendicular.

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12 (b) Let A and B be $n \times n$ invertible matrices. Show that if $A+B$ is invertible, then $A^{-1} + B^{-1}$ is also invertible.
(Hint: Consider the product $A^{-1}(A + B)\, B^{-1}$.)

A and B are invertible matrices $\Rightarrow$ A^{-1} and B^{-1} exist.

$$A^{-1}(A+B)B^{-1} = \left(A^{-1}A + A^{-1}B\right)B^{-1} = IB^{-1} + A^{-1}BB^{-1} = B^{-1} + A^{-1}$$

Method 1:

$\Rightarrow$ $\det\left(A^{-1} + B^{-1}\right) = \det\left(A^{-1}(A+B)B^{-1}\right) = \det A^{-1} \cdot \det(A+B) \cdot \det B^{-1}$
But all three matrices A^{-1}, B^{-1} and $(A+B)$ are invertible
$\Rightarrow$ $\det A^{-1}$, $\det(A+B)$ and $\det B^{-1}$ are all non-zero
$\Rightarrow$ $\det A^{-1} \cdot \det(A+B) \cdot \det B^{-1} \neq 0$ $\Rightarrow$ $\det\left(A^{-1} + B^{-1}\right) \neq 0$
Therefore $(A^{-1} + B^{-1})$ is also invertible.

Method 2:

$(A+B)$ is invertible $\Rightarrow$ $(A+B)^{-1}$ exists

$\left(A^{-1}(A+B)B^{-1}\right)^{-1} = \left(B^{-1}\right)^{-1}(A+B)^{-1}\left(A^{-1}\right)^{-1} = B(A+B)^{-1}A$ clearly exists
But $\left(A^{-1}(A+B)B^{-1}\right)^{-1} = \left(A^{-1} + B^{-1}\right)^{-1}$
Therefore $(A^{-1} + B^{-1})$ is also invertible.

Note that a general proof is required - it is ***not*** sufficient to show that $(A^{-1} + B^{-1})$ is invertible for some particular choice of matrices A and B.

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