ENGI 3424
Tutorial Example for Series

Find the interval of convergence I for the series

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{4}{x-3}\right)^{n/3}$$

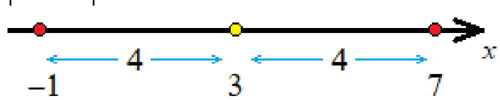
Note that this is *not* a standard Taylor series, unless one adopts the change of variables $z = \left(\frac{4}{x-3}\right)^{1/3}$.

Ratio test:

$$\frac{u_{n+1}}{u_n} = \frac{1}{n+1}\left(\frac{4}{x-3}\right)^{\frac{n+1}{3}} \cdot \frac{n}{1}\left(\frac{4}{x-3}\right)^{-\frac{n}{3}} = \frac{n}{n+1}\left(\frac{4}{x-3}\right)^{\frac{1}{3}}$$

$$\Rightarrow \lim_{n\to\infty}\left(\left|\frac{u_{n+1}}{u_n}\right|\right) = \left|\frac{4}{x-3}\right|^{\frac{1}{3}} \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right) = \left|\frac{4}{x-3}\right|^{\frac{1}{3}}$$

$$\lim_{n\to\infty}\left(\left|\frac{u_{n+1}}{u_n}\right|\right) < 1 \quad \text{if} \quad \left(\frac{4}{|x-3|}\right)^{1/3} < 1 \Rightarrow \frac{4}{|x-3|} < 1 \Rightarrow |x-3| > 4$$



$$\Rightarrow \quad x-3 > 4 \quad \text{or} \quad x-3 < -4$$

$$\Rightarrow \quad x > 4+3 = 7 \quad \text{or} \quad x < -4+3 = -1$$

This series converges absolutely for $x < -1$ or $x > 7$ and diverges for $-1 < x < 7$.

Checking the endpoints:

$$x = -1 \Rightarrow u_n = \frac{1}{n}\left(\frac{4}{-1-3}\right)^{n/3} = \frac{1}{n}\left(\sqrt[3]{-1}\right)^n = \frac{(-1)^n}{n}$$

At $x = -1$ we have the alternating harmonic series, which is conditionally convergent.

$$x = 7 \Rightarrow u_n = \frac{1}{n}\left(\frac{4}{7-3}\right)^{n/3} = \frac{1}{n}(1)^n = \frac{1}{n}$$

At $x = 7$ we have the harmonic series, which is divergent.

Therefore the interval of convergence is

$$\boxed{I = (-\infty, -1] \cup (7, \infty)}$$

with conditional convergence at $x = -1$.
