# 6. Series

**Contents:**



**Appendix:**

## 6.01 Sequences; general term, limits, convergence

A sequence is a set of related objects that all follow the same rule. There is a logical progression from current and previous elements to the next element.

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Example 6.01.1

The alphabet: { a, b, c, ... , y, z }

The rule is: The $n^{\text{th}}$ element $a_n$ = (the $n^{\text{th}}$ letter of the alphabet)

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Example 6.01.2

The set of all natural numbers $\mathbb{N}$ = { 1, 2, 3, ... }

Two ways to express the rule are:

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We are usually interested in two features:

- An explicit form for the general term (often from a recurrence relationship);
- The behaviour of the terms of a sequence as $n \to \infty$.

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Example 6.01.3

The terms of a sequence form an **arithmetic progression** if consecutive terms differ by a common non-zero constant difference $d$ :

$a_n - a_{n-1} = d \quad \Rightarrow \quad a_n = a_{n-1} + d$ , with an initial term $a_1 = a$

Writing down the first few terms of the sequence,

The general term is therefore $a_n =$

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<u>Example 6.01.4</u>

The terms of a sequence form a **geometric progression** if consecutive terms change by a common constant ratio $r$ :

$$\frac{a_n}{a_{n-1}} = r \quad \Rightarrow \quad a_n = a_{n-1} r \qquad (r \neq 0,\ r \neq 1)\text{, with an initial term } \quad a_1 = a \neq 0$$

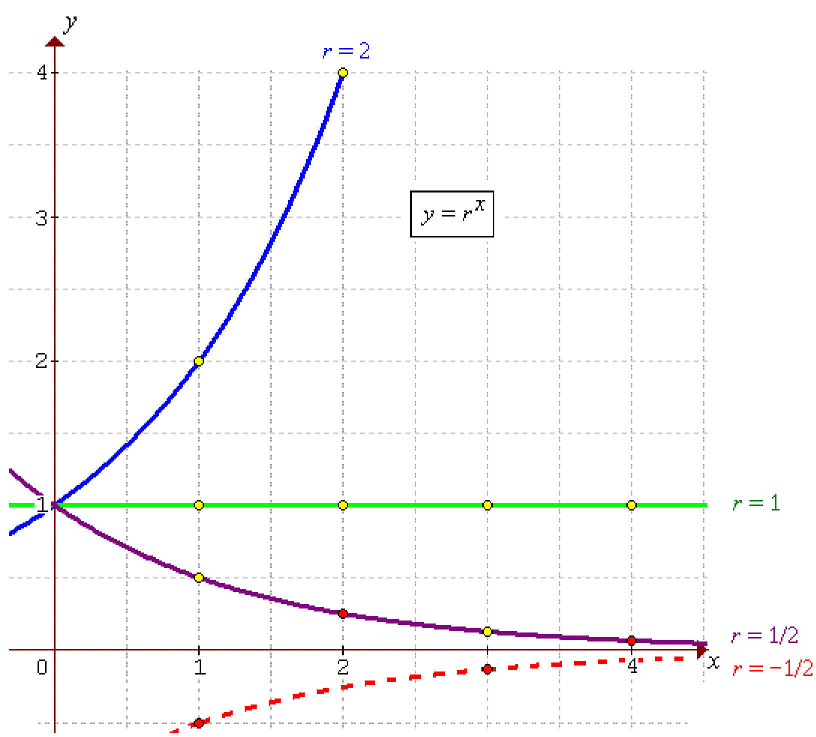
Writing down the first few terms of the sequence,

The general term is therefore $a_n =$

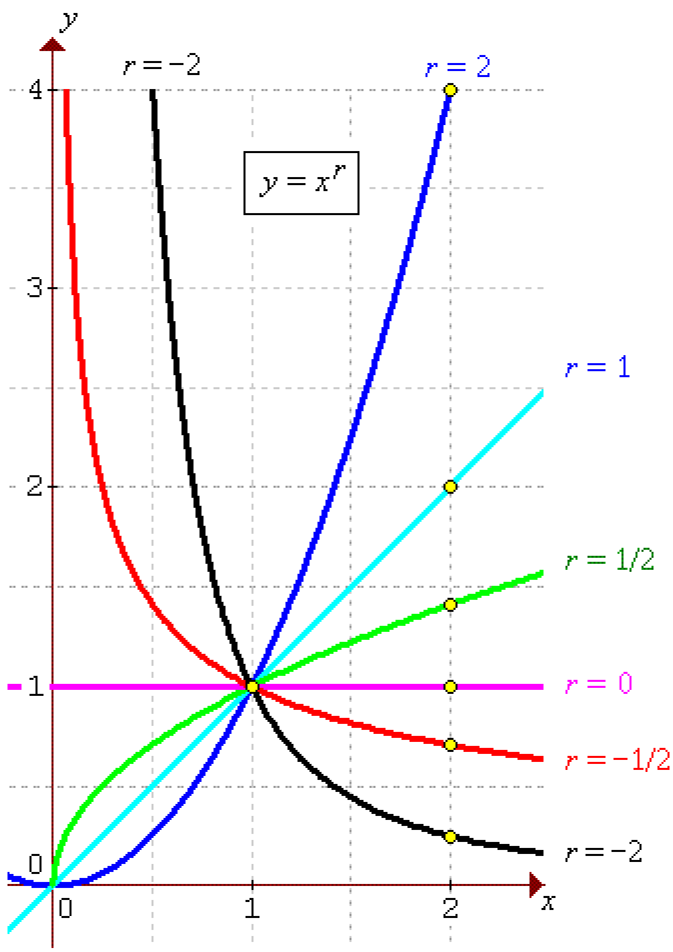
The sequence alternates in sign if $r < 0$

The sequence converges to a limit of zero if

If $r = -1$ then $\lim\limits_{n \to \infty} a_n$



The power sequence $\left\{ n^r \right\} = \left\{ 1^r, 2^r, 3^r, 4^r, \ldots \right\}$
converges to 0 for $r < 0$,
converges to 1 for $r = 0$ and
diverges for $r > 0$.



<u>Example 6.01.5</u>

Find the general term of this sequence and determine whether it converges.

$$\left\{ \frac{1}{5},\ \frac{4}{9},\ \frac{7}{13},\ \frac{10}{17},\ \frac{13}{21},\ \ldots \right\}$$

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**Algebra of Sequences:**

Given two convergent sequences $\{ a_n \}$ with limit value $A$ and $\{ b_n \}$ with limit value $B$,
$\{ pa_n + qb_n \}$ is a convergent sequence with limit value $(pA + qB)$ for all constants $p$, $q$;
$\{ a_n b_n \}$ is a convergent sequence with limit value $AB$;
$\{ a_n / b_n \}$ is a convergent sequence with limit value $A/B$ (provided that $B \neq 0$);
If $f(x)$ is a continuous function with $\lim\limits_{n \to \infty} f(x) = L$ and if $a_n = f(n) \quad \forall n$
then $\{ a_n \}$ converges with limit value $L$;
If $a_n \le c_n \le b_n \quad \forall n$ and if $\{ c_n \}$ converges with limit value $C$ then $A \le C \le B$.

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Example 6.01.6

Find the limit of the sequence $\left\{ \dfrac{\sin n}{n} \right\}$.

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**The "Race to Infinity"**

There is a hierarchy of functional forms that all diverge to infinity with increasing $n$:

$$n^n \quad > \quad n! \quad > \quad a^n \ \left(a > 1\right) \quad > \quad n^c \ \left(c > 0\right) \quad > \quad \log_b n \ \left(b > 1\right)$$

for all sufficiently large $n$.

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Example 6.01.7

$\lim\limits_{n \to \infty} \dfrac{n!}{n^n} = 0$ because "$n^n$ goes to infinity faster than $n\,!$"

OR the sequence is positive and

$$\frac{n!}{n^n} = \frac{n\left(n-1\right)\left(n-2\right)\ldots\times 3\times 2\times 1}{n\times n\times n\times\ldots\times n\times n\times n} < \frac{n}{n}\cdot\frac{n}{n}\cdot\frac{n}{n}\cdot\cdots\cdot\frac{n}{n}\cdot\frac{n}{n}\cdot\frac{1}{n} = \frac{1}{n} \quad \forall n > 1$$

Therefore all terms are bounded above by the terms of a sequence that converges to 0.

---

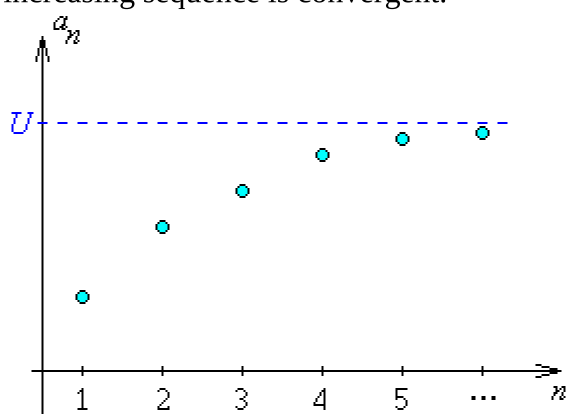
If every term of a sequence is no less than all of its predecessors $(a_1 \le a_2 \le a_3 \le a_4 \le \dots)$, then the sequence is an **increasing sequence**.

If every term of a sequence is no greater than all of its predecessors $(a_1 \ge a_2 \ge a_3 \ge \dots)$, then the sequence is a **decreasing sequence**.

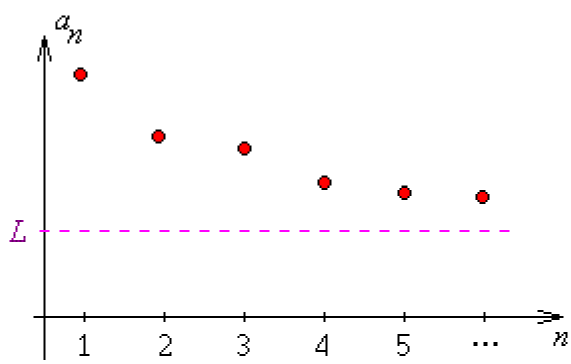
A sequence is **monotonic** if and only if it is either increasing or decreasing.

If $a_n \le U \quad \forall n$ for some constant value $U$, then the sequence $\{ a_n \}$ is **upper-bounded**.
If $a_n \ge L \quad \forall n$ for some constant value $L$, then the sequence $\{ a_n \}$ is **lower-bounded**.
A sequence is **bounded** if it is both upper and lower bounded.

Every upper-bounded increasing sequence is convergent.




Every lower-bounded decreasing sequence is convergent.




**Every bounded monotonic sequence is convergent.**

## 6.02 Series

The partial sum $S_n$ of a sequence is the sum of the first $n$ terms of that sequence:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

This partial sum is also a finite series.
The sum of an infinite series is the sum of all terms in the associated infinite sequence,

$$S = a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n .$$

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Example 6.02.1

Find the first three partial sums of the sequence $\left\{ \frac{1}{2^n} \right\}, \quad (n = 1, 2, 3, \ldots)$ and find the limit of the sequence of partial sums.

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**Divergence Test**

If the general term of a series does not converge to zero, then the series diverges.

$$\lim_{n\to\infty} a_n \neq 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n \text{ diverges (the sum does not exist)}$$

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Example 6.02.2

Prove that all arithmetic series $\sum_{n=1}^{\infty}\left(a+(n-1)d\right)$ diverge, (where $d \neq 0$).

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However, the converse of the divergence test is false.
Some divergent series do have general terms that converge to zero,

or equivalently: $\lim_{n\to\infty} a_n = 0$ does ***not*** guarantee that $\sum_{n=1}^{\infty} a_n$ converges

(except for alternating series, section 6.05 below).

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Example 6.02.3

Show that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

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The divergence of the harmonic series is *very* slow.
4 terms are needed for the partial sum to exceed 2.
31 terms are needed for the partial sum to exceed 4.
1674 terms are needed for the partial sum to exceed 8.
The sum of the first *billion* ($10^9$) terms is still well under 25.
Yet the sum of the complete series does not exist (diverges to infinity)!

## 6.03 <u>Standard Series</u>: telescoping series, geometric series, *p*-series

<u>Example 6.03.1</u>

Find the exact sum of the series $S = \sum_{n=1}^{\infty} \frac{1}{n^2 + 5n + 6}$ .

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This is an example of a telescoping series.
The simplest such series have general terms similar to $a_n = f(n) - f(n+1)$.
Such series often involve the use of partial fractions.

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Example 6.03.2

Find the exact sum of the series $S = \sum_{n=2}^{\infty} \frac{8}{n^3 + 3n^2 - n - 3}$ .

Example 6.03.2 (continued)

$$
\begin{array}{lllll}
S_n = & \frac{1}{1} - \frac{2}{3} + \frac{1}{5} & & & \\
& \frac{1}{2} - \frac{2}{4} + \frac{1}{6} & & & \\
& \frac{1}{3} - \frac{2}{5} + \frac{1}{7} & & & \frac{1}{n-5} - \frac{2}{n-3} + \frac{1}{n-1} \\
& \frac{1}{4} - \frac{2}{6} + \frac{1}{8} & & & \frac{1}{n-4} - \frac{2}{n-2} + \frac{1}{n} \\
& \frac{1}{5} - \frac{2}{7} + \frac{1}{9} & & \cdots & \frac{1}{n-3} - \frac{2}{n-1} + \frac{1}{n+1} \\
& \frac{1}{6} - \frac{2}{8} + \frac{1}{10} & & & \frac{1}{n-2} - \frac{2}{n} + \frac{1}{n+2} \\
& \frac{1}{7} - \frac{2}{9} + \frac{1}{11} & & & \frac{1}{n-1} - \frac{2}{n+1} + \frac{1}{n+3}
\end{array}
$$

Collecting the surviving terms from this telescoping series, we find that

## Geometric Series

Each term in a geometric series is a constant multiple $r$ (the common ratio) of the immediately preceding term, except for the [non-zero] first term $a$. Quoting the general term from the geometric progression on page 6.03, the geometric series is

$$S = \sum_{n=1}^{\infty} a r^{n-1} \quad \left( \text{or} \quad \sum_{n=0}^{\infty} a r^{n} \right)$$

Applying the divergence test:

Example 6.03.3 (Example 6.02.1 revisited)

Evaluate $S = \sum_{n=1}^{\infty} \frac{1}{2^n}$ .

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Example 6.03.4

Find the sum of the power series

$$f(x) = x - x^2 + x^3 - x^4 + \dots$$

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***p*-series**

Another important family of standard series is the *p*-series
(also known as "hyperharmonic" series):

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

The divergence test establishes divergence for $p \le 0$ .

It can be shown [section 6.A] that *p*-series

- converge for all $p > 1$ and
- diverge for all $p \le 1$ .

The case $p = 1$ is the harmonic series, for which Example 6.02.3 established divergence.

In 1741 Leonhard Euler proved that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

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The removal or insertion of a finite number of finite terms does *not* affect the convergence of a series.

If $S = \displaystyle\sum_{n=3}^{\infty} a_n$ converges, then so does $\displaystyle\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \sum_{n=3}^{\infty} a_n = a_1 + a_2 + S$ ,

provided that $a_1$ and $a_2$ are finite.

## 6.04 <u>Tests for Convergence</u>: comparison and limit comparison tests

If all terms $a_n$ in a series are positive, then the sequence of partial sums $\{S_n\}$ is necessarily increasing.

If another series $\sum_{n=1}^{\infty} b_n$ is convergent with a sum $B$ and if $a_n \le b_n \quad \forall n$, then $B$ is an upper bound to $\{S_n\}$

Recall that an increasing sequence that has an upper bound is convergent [page 6.07]

$\Rightarrow \; \{S_n\}$ must converge to a value $\le B$

$\Rightarrow \; \sum_{n=1}^{\infty} a_n$ converges to some value $A \le B$.

This is the basis of the **comparison test**.

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<u>Example 6.04.1</u>

Is the series $\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6}$ convergent?

[This is Example 6.03.1, for which the exact value of the sum was found by telescoping the series.]

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Example 6.04.2

Is the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ convergent?

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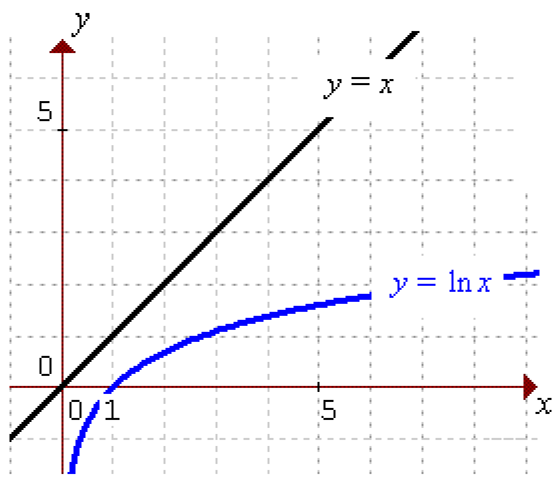


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Put very briefly, the essence of the comparison test is:
A convergent ceiling forces convergence:

$$0 \le a_n \le b_n \quad \forall n \quad \text{and} \quad \sum b_n \text{ converges} \Rightarrow \quad \sum a_n \text{ converges.}$$

A divergent floor forces divergence:

$$a_n \ge b_n \ge 0 \quad \forall n \quad \text{and} \quad \sum b_n \text{ diverges} \Rightarrow \quad \sum a_n \text{ diverges.}$$

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## Limit Comparison Test

All terms $a_n$ and $b_n$ must be non-negative and $L$ is some finite constant.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \geq 0 \quad \text{and} \quad \sum b_n \text{ converges} \Rightarrow \sum a_n \text{ converges.}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} \neq 0 \quad \text{and} \quad \sum b_n \text{ diverges} \Rightarrow \sum a_n \text{ diverges.}$$

The reference series $\sum_{n=1}^{\infty} b_n$ is often the geometric series or a *p*-series.

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Example 6.04.3

Is the series $\sum_{n=3}^{\infty} \frac{n \ln n}{n^3 - 8}$ convergent?

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Example 6.04.4

Is the series $\sum_{n=1}^{\infty} \frac{n^2-4}{n^3+9n}$ convergent?

## 6.05 Tests for Convergence: alternating series; absolute and conditional convergence

### Alternating Series Test

If consecutive terms of a series alternate in sign, then a simpler test for convergence is available.

$$\text{If } \frac{a_{n+1}}{a_n} < 0 \quad \forall n \quad \text{and} \quad \lim_{n\to\infty} a_n = 0 \quad \text{then} \quad \sum_{n=1}^{\infty} a_n \quad \text{converges.}$$

---

Example 6.05.1

Is the alternating harmonic series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

convergent?

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Note that if we take the absolute values of the terms in the alternating harmonic series, then we obtain the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$, which diverges.

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**Absolute Convergence**

If and only if the series $\sum_{n=1}^{\infty} |a_n|$ converges,

then the series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent**.

If and only if the series $\sum_{n=1}^{\infty} a_n$ converges but the series $\sum_{n=1}^{\infty} |a_n|$ diverges,

then the series $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent**.
The alternating harmonic series is an example of a conditionally convergent series.

If a series is absolutely convergent, then the alternating series test is a waste of time. Another test to establish the convergence of $\sum_{n=1}^{\infty} |a_n|$ automatically establishes that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent in one step.

If a series is conditionally convergent, then the alternating series test is often required to establish convergence of $\sum_{n=1}^{\infty} a_n$, but another test is required on $\sum_{n=1}^{\infty} |a_n|$ to complete the proof of conditional convergence. Therefore test the convergence of $\sum_{n=1}^{\infty} |a_n|$ first!

---

Absolute convergence allows the algebra of finite series to be extended to infinite series. In particular, the order of an infinite number of terms may be changed without affecting the sum of the series. The derivative of an absolutely convergent series is the sum of the derivatives of the individual terms.

This is no longer necessarily true for conditionally convergent series. For example,
$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots = \ln 2$ but $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots = \frac{3}{2} \ln 2$.

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Example 6.05.2

Investigate the convergence of the alternating *p*-series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots$$

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This is obviously an alternating series.

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Example 6.05.3

Is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$ absolutely convergent, conditionally convergent or divergent?

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## Approximation Errors

For any convergent alternating series, the error in approximating the sum $S$ of the entire series caused by evaluating the partial sum $S_n$ of just the first $n$ terms is

$$| S - S_n | \le | a_{n+1} |$$

---

Example 6.05.4

Estimate $S = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$ correct to four decimal places.

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The signs of the terms of this series are clearly alternating and $\lim_{n\to\infty} \frac{(-1)^n}{(2n)!} = 0$

$\Rightarrow$ this series converges (by the alternating series test).

The error caused by using $S_n$ to estimate $S$ is

## 6.06 Tests for Convergence: ratio test

Among the most useful tests for series convergence is the **ratio test**:

If $\displaystyle \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\displaystyle \sum_{n=1}^{\infty} a_n$ is absolutely convergent.

If $\displaystyle \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ then $\displaystyle \sum_{n=1}^{\infty} a_n$ is divergent.

If $\displaystyle \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ then the test fails and there is no information on series convergence.

The ratio test is most useful when the general term includes an exponential or factorial factor. The test fails when the general term is an algebraic function (terms of the form $n^{\text{constant}}$ in the numerator and/or denominator) only.

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Example 6.06.1

Is the series $\displaystyle \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^n}{n!}$ absolutely convergent, conditionally convergent or divergent?

Note:

Proof that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$:

Example 6.06.2

Is the series $\displaystyle\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ absolutely convergent, conditionally convergent or divergent?

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There is a test for absolute convergence known as the "root test", which is a close relative of the ratio test. However, we shall not use the root test in ENGI 3425.

## 6.07 <u>Power series</u>, radius and interval of convergence

A power series with centre $x = c$ is of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^{bn+d} = (x-c)^d \left( a_0 + a_1 (x-c)^b + a_2 (x-c)^{2b} + a_3 (x-c)^{3b} + \ldots \right)$$

where $b$ and $d$ are real constants and $b \neq 0$.

A common choice of parameters for power series is $b = 1$ and $d = 0$:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + a_3 (x-c)^3 + \ldots$$

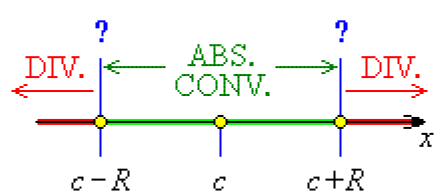
Let $u_n$ now represent the general term $a_n (x-c)^{bn+d}$ and
let us apply the ratio test for convergence of a power series:

$$\frac{u_{n+1}}{u_n} = \frac{a_{n+1} (x-c)^{bn+b+d}}{a_n (x-c)^{bn+d}} = \frac{a_{n+1} (x-c)^b}{a_n}$$

$$\Rightarrow \quad \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \left| (x-c)^b \right| \cdot \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

The series is absolutely convergent whenever

where $R$ is the **radius of convergence**




The interval of convergence $I$ includes $c - R < x < c + R$ and may include one or both endpoints, which must be tested separately.
$I$ is also the domain of the function $f(x)$.

Also note that $f(c) = a_0$ .
All power series are absolutely convergent at the centre, even if $R = 0$.
If $R = \infty$, then the power series is absolutely convergent for all $x$.

Example 6.07.1

Determine the radius of convergence and interval of convergence for the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(x+2)^n}{n+3}$$

Example 6.07.2

Determine the radius of convergence and interval of convergence for the power series

$$f\left(x\right) = \sum_{n=1}^{\infty} \frac{\left(x-1\right)^n}{3^n n\sqrt{n}}$$

## 6.08 Taylor and Maclaurin Series

A function $f(x)$ that is represented by a power series (using only non-negative integer powers of $(x-c)$) is a Taylor series:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

$$f(c) = a_0 + 0 + 0 + \ldots \quad \Rightarrow \quad a_0 = f(c)$$

Assuming that the series exists and is absolutely convergent, then

$$f'(x) = \sum_{n=0}^{\infty} a_n\, n(x-c)^{n-1}$$

$$\Rightarrow \quad f'(c) = 0 + 1a_1 + 0 + 0 + \ldots \quad \Rightarrow \quad a_1 = \frac{f'(c)}{1}$$

$$f''(x) = \sum_{n=0}^{\infty} a_n\, n(n-1)(x-c)^{n-2}$$

$$\Rightarrow \quad f''(c) = 0 + 0 + 2\times 1 a_2 + 0 + \ldots \quad \Rightarrow \quad a_2 = \frac{f''(c)}{2\times 1}$$

$$f'''(x) = \sum_{n=0}^{\infty} a_n\, n(n-1)(n-2)(x-c)^{n-3}$$

$$\Rightarrow \quad f'''(c) = 0 + 0 + 0 + 3\times 2\times 1 a_3 + \ldots \quad \Rightarrow \quad a_3 = \frac{f'''(c)}{3\times 2\times 1}$$

and so on, so that the general term is $a_n = \dfrac{f^{(n)}(c)}{n!}$ and the Taylor series for $f(x)$ is

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n}$$

The ratio test can determine the radius of convergence $R$.

Example 6.08.1

Find the Taylor series for $f(x) = \ln x$ about $x = 1$.

Example 6.08.2

Prove L'Hôpital's rule for the case of a (0/0) indeterminacy.

Let functions $f(x)$ and $g(x)$ be such that $f(a) = g(a) = 0$.
Then $\dfrac{f(a)}{g(a)}$ is a $\dfrac{0}{0}$ type of indeterminacy.
Represent both functions by their Taylor series about $x = a$, then

## Maclaurin Series

A Taylor series with a centre at $x = c = 0$ is a Maclaurin series.

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Example 6.08.3

Find the Maclaurin series for $f(x) = e^x$ and find its interval of convergence.

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Example 6.08.4

Find the Maclaurin series for $f(x) = \cos x$ and find its interval of convergence.

Example 6.08.4 (Alternative Solution)

Use the Euler expression for the cosine function in terms of the exponential function:

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$e^{j\theta} = \cos\theta + j\sin\theta$ and

$\Rightarrow \quad \cos\theta =$

<u>Example 6.08.5</u>

Use the Maclaurin series for the cosine function to find the Maclaurin series for the sine function.

---

The Maclaurin series for $f(x) = \sin x$ may also be found directly, through repeated differentiations, or via the Euler equation and the Maclaurin series for the exponential function (as in Example 6.08.4 for the cosine function) or by differentiation of $-\cos x$ .

**Remainder Term**

The practical use of Taylor (and Maclaurin) series arises when the partial sum of the first few terms of the series $T_n(x)$ is used to approximate the value of a non-linear function. The error caused by truncating the series at the $n^{\text{th}}$ term is the $n^{\text{th}}$ remainder term $R_n(x)$:

$$R_n(x) = \left| f(x) - T_n(x) \right| = \frac{f^{(n+1)}(\xi)}{(n+1)!} |x-c|^{n+1}$$

where $\xi$ is some number between $x$ and $c$.

The Taylor series is well-defined only if $\lim_{n\to\infty} R_n(x) = 0 \qquad (c-R < x < c+R)$.

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Example 6.08.6

The Maclaurin series for the exponential function $f(x) = e^x$ is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} |x-0|^{n+1} = \frac{e^{\xi} x^{n+1}}{(n+1)!} \qquad (0 < |\xi| < |x|)$$

Because factorial functions diverge to infinity faster than exponential functions,

$$\lim_{n\to\infty} R_n(x) = e^{\xi} \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

and the Maclaurin series is therefore well-defined.

The error caused by approximating $f(x) = e^x$ by the quadratic $T_2(x) = 1 + x + \frac{x^2}{2!}$ on the interval $(0 < x < a)$ is at most $\frac{a^3 e^a}{6}$.

For $a = 0.1$, this maximum error is less than $2\times10^{-4}$, with a maximum relative error of under 0.03%.

However, for $a = 10$, the maximum error is substantial: more than $3\times10^{6}$. Many more terms need to be taken to maintain accuracy, the further away from the centre one goes.

Note that $y = T_1(x)$ is the equation of the tangent line to $y = f(x)$ at $x = c$.

## 6.09 Binomial Series

A special case of Maclaurin series arises for $f(x) = (1+x)^n$.

Summary:

$$\boxed{(1+x)^n = 1 + \sum_{k=1}^{\infty} \frac{n(n-1)(n-2)\ldots(n-(k-1))}{k!}\,x^k}$$

Interval of convergence:
$I = \mathbb{R}$ for $n = 0$ or $n \in \mathbb{N}$ [ the series terminates at $k = n$ ]
$I = [-1, 1]$ for $n > 0$ and not an integer, with absolute convergence at $x = \pm 1$.
$I = (-1, 1]$ for $-1 < n < 0$ with conditional convergence at $x = +1$.
$I = (-1, 1)$ for $n \le -1$ with divergence at $x = \pm 1$.

<u>Example 6.09.1</u>

Find the Maclaurin series for $f(x) = \sqrt{1+2x}$ and its interval of convergence.

---

In the binomial expansion, for "$x$" read "$2x$" and for "$n$" read "½".

## 6.10 Fourier Series

This section offers a very brief overview of [discrete] Fourier series. Some majors will explore Fourier series and transforms in much greater detail in subsequent semesters.

---

The **Fourier series** of $f(x)$ on the interval $(-L, L)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad (n = 0, 1, 2, 3, \ldots)$$

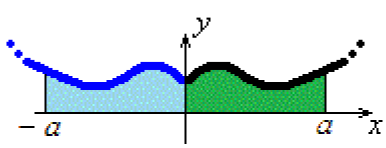
and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad (n = 1, 2, 3, \ldots)$$

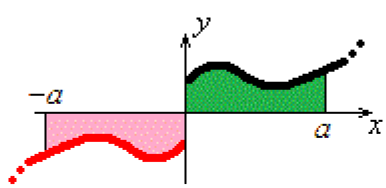
---

The $\{a_n, b_n\}$ are the Fourier coefficients of $f(x)$.

Note that the cosine functions (and the function 1) are even, while the sine functions are odd.




If $f(x)$ is even ($f(-x) = +f(x)$ for all $x$), then $b_n = 0$ for all $n$, leaving a Fourier cosine series (and perhaps a constant term) only for $f(x)$.




If $f(x)$ is odd ($f(-x) = -f(x)$ for all $x$), then $a_n = 0$ for all $n$, leaving a Fourier sine series only for $f(x)$.

Note that
$$\int_{-a}^{a} g(x)\,dx = \begin{cases} 0 & \text{if } g(x) \text{ is odd} \\ 2\int_0^a g(x)\,dx & \text{if } g(x) \text{ is even} \end{cases}$$

with the multiplication table

| × | odd function | even function |
|---|---|---|
| odd function | even | odd |
| even function | odd | even |

Example 6.10.1

Expand $$f(x) = \begin{cases} 0 & (-\pi < x < 0) \\ \pi - x & (0 \le x < +\pi) \end{cases}$$ in a Fourier series.

Example 6.10.1 (Additional Notes – also see "`www.engr.mun.ca/~ggeorge/3425/demos/`")

The first few partial sums in the Fourier series

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right) \qquad (-\pi < x < +\pi)$$

are

$$S_0 = \frac{\pi}{4}$$

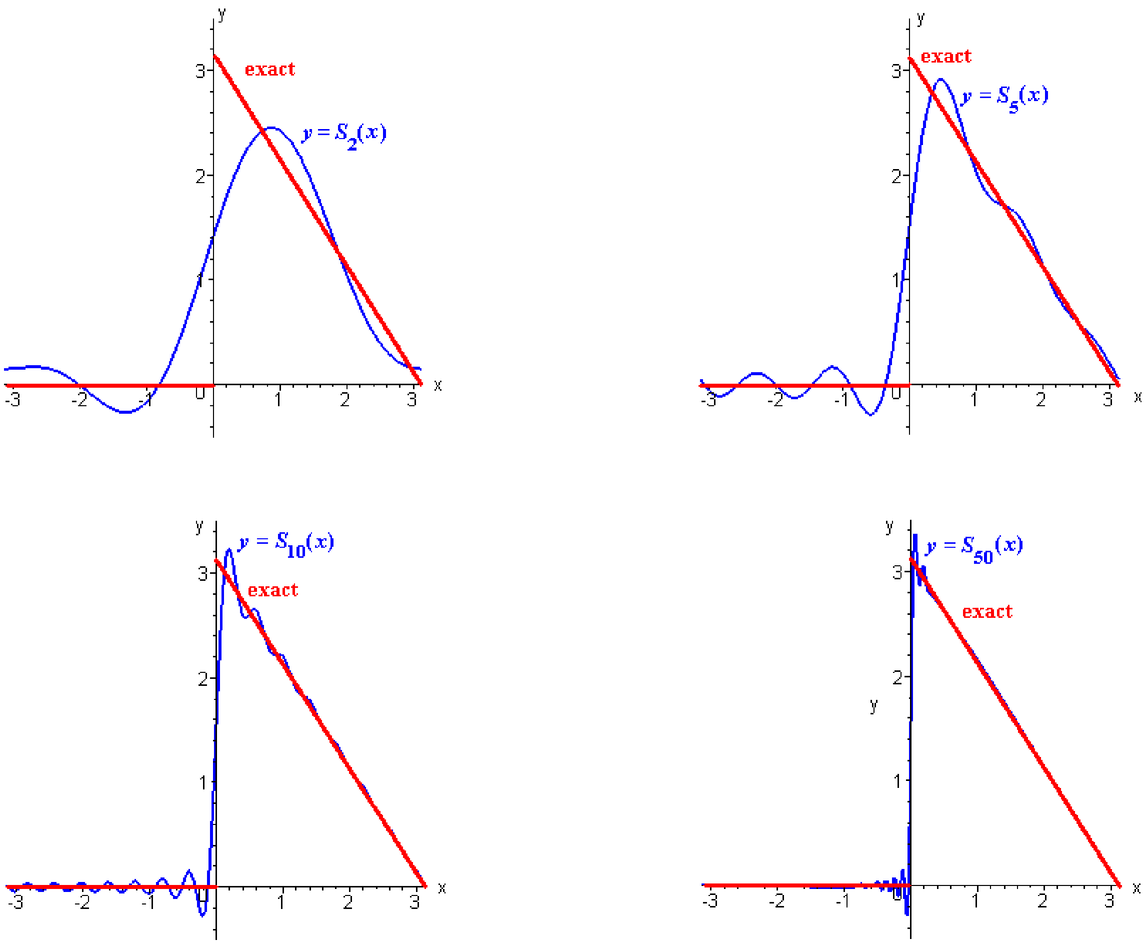
$$S_1 = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x$$

$$S_2 = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x$$

$$S_3 = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x + \frac{2}{9\pi}\cos 3x + \frac{1}{3}\sin 3x$$

and so on.

The graphs of successive partial sums approach $f(x)$ more closely, except in the vicinity of any discontinuities, (where a systematic overshoot occurs, the **Gibbs phenomenon**).



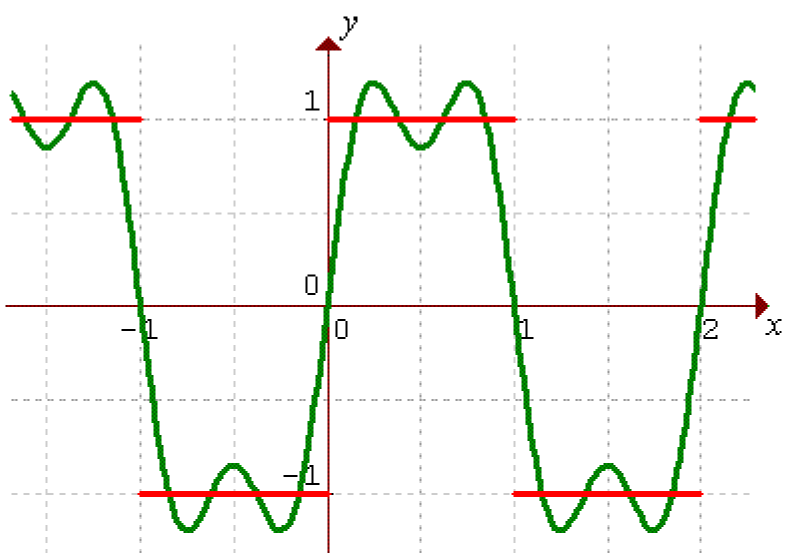
Example 6.10.2

Find the Fourier series expansion for the standard square wave,

$$f(x) = \begin{cases} -1 & (-1 < x < 0) \\ +1 & (0 \le x < +1) \end{cases}$$

---

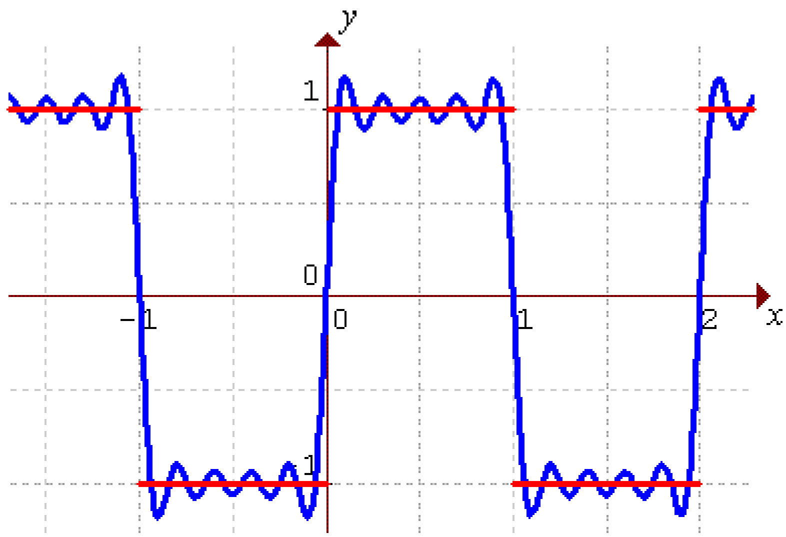
The graphs of the third and ninth partial sums (containing two and five non-zero terms respectively) are displayed here, together with the exact form for $f(x)$, with a **periodic extension** beyond the interval (–1, +1) that is appropriate for the square wave.




$y = S_3(x)$

Example 6.10.2 (continued)

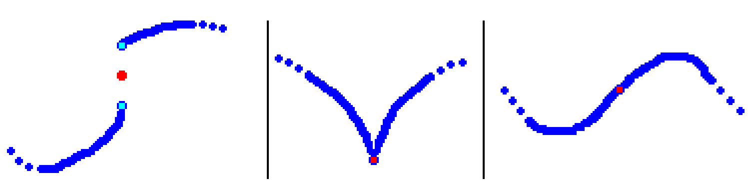
$$y = S_9(x)$$




## Convergence

At all points $x = x_o$ in $(-L, L)$ where $f(x)$ is continuous and is either differentiable or the limits $\lim_{x \to x_o^-} f'(x)$ and $\lim_{x \to x_o^+} f'(x)$ both exist, the Fourier series converges to $f(x)$.

At finite discontinuities, (where the limits $\lim_{x \to x_o^-} f'(x)$ and $\lim_{x \to x_o^+} f'(x)$ both exist), the Fourier series converges to $\dfrac{f(x_o -) + f(x_o +)}{2}$,
(using the abbreviations $f(x_o -) = \lim_{x \to x_o^-} f(x)$ and $f(x_o +) = \lim_{x \to x_o^+} f(x)$).



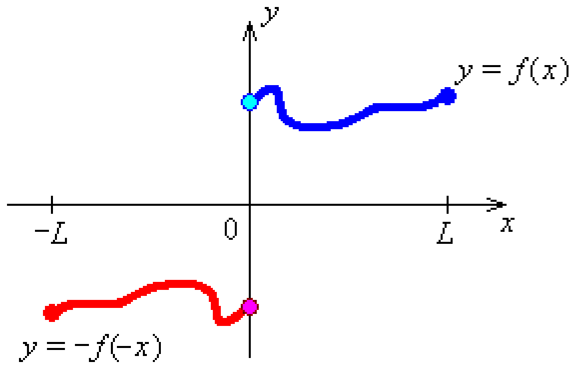
$f(x)$ not continuous at $x = x_o$ | continuous but not differentiable | continuous and differentiable

In all cases, the Fourier series at $x = x_o$ converges to $\dfrac{f(x_o -) + f(x_o +)}{2}$ (the red dot).

## Half-Range Fourier Series

A Fourier series for $f(x)$, valid on $[0, L]$, may be constructed by extension of the domain to $[-L, L]$.

An odd extension leads to a **Fourier sine series**:



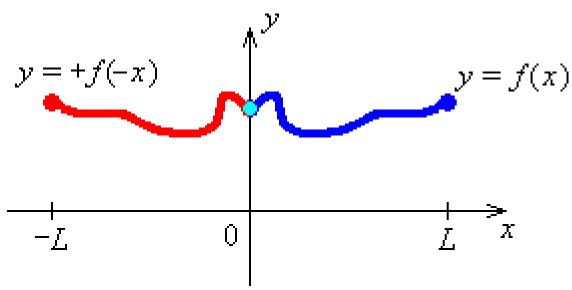

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\,, \quad (n = 1, 2, 3, \ldots)$$

---

An even extension leads to a **Fourier cosine series**:




$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\,, \quad (n = 0, 1, 2, 3, \ldots)$$

and there is automatic continuity of the Fourier cosine series at $x = 0$ and at $x = \pm L$.

---

Example 6.10.3

Find the Fourier sine series and the Fourier cosine series for $f(x) = x$ on [0, 1].

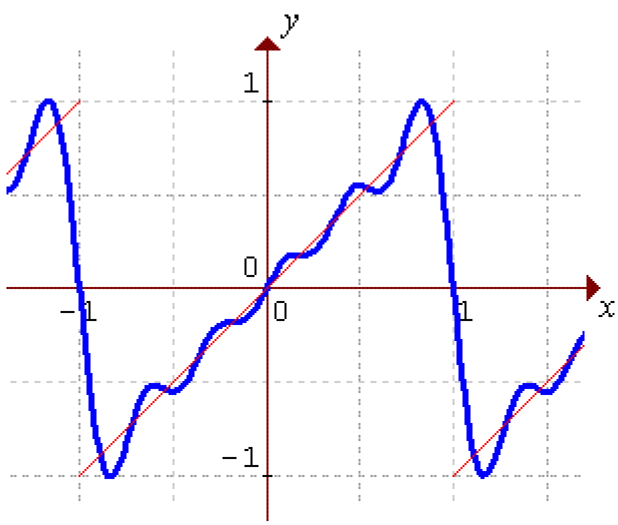
---

$f(x) = x$ happens to be an odd function of $x$ for any domain centred on $x = 0$. The odd extension of $f(x)$ to the interval $[-1, 1]$ is $f(x)$ itself.

Evaluating the Fourier sine coefficients,

This function happens to be continuous and differentiable at $x = 0$, but is clearly discontinuous at the endpoints of the interval ($x = \pm 1$).

Fifth order partial sum of the Fourier sine series for $f(x) = x$ on [0, 1]



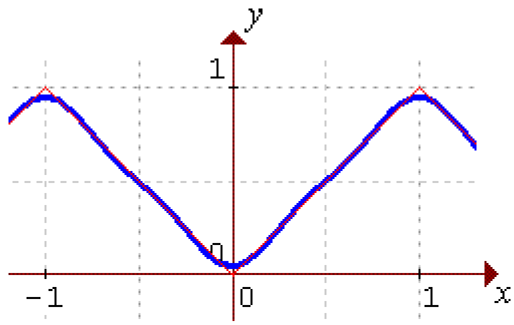
Example 6.10.3 (continued)

The even extension of $f(x)$ to the interval $[-1, 1]$ is $f(x) = |x|$.

Evaluating the Fourier cosine coefficients,

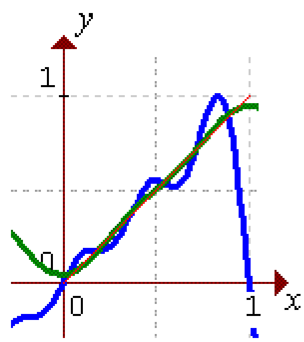
Example 6.10.3 (continued)

Third order partial sum of the Fourier cosine series for $f(x) = x$ on [0, 1]




Note how rapid the convergence is for the cosine series compared to the sine series.

$y = S_3(x)$ for cosine series and $y = S_5(x)$ for sine series for $f(x) = x$ on [0, 1]



## 6.A Integral Test [for reference only – not examinable in this course]

If $f(x)$ is a continuous function with positive values that are decreasing for all $x \geq k$, $a_n = f(n) \quad (n = k,\ k+1,\ k+2, \ldots)$ and the improper integral $\lim\limits_{L\to\infty} \int_{x=k}^{L} f(x)\,dx$ exists as a finite real number, then the series $\sum\limits_{n=k}^{\infty} a_n$ converges and

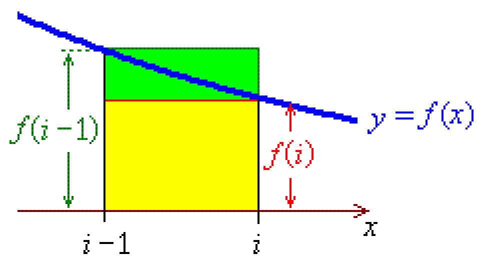
$$\boxed{\int_k^\infty f(x)\,dx \leq \sum_{n=k}^{\infty} a_n \leq a_k + \int_k^\infty f(x)\,dx}$$

If the improper integral does not have a finite value, then the series diverges.

---

**Proof:**

Examine the area between $x = i - 1$ and $x = i$ under the curve $y = f(x)$:




Both rectangles have width one unit.
The smaller rectangle has height $f(i)$ and area smaller than that under the curve.
The larger rectangle has height $f(i-1)$ and area larger than that under the curve.
Therefore

$$1\times f(i) < \int_{i-1}^{i} f(x)\,dx < 1\times f(i-1)$$

This inequality is true for all areas from $i - 1 = k$ onwards.
Adding all of these areas together, we find

$$\sum_{i=k+1}^{\infty} f(i) < \int_k^\infty f(x)\,dx < \sum_{i=k+1}^{\infty} f(i-1)$$

The left side can be re-arranged as

$$\int_k^\infty f(x)\,dx > \sum_{i=k}^{\infty} f(i) - f(k) \quad \Rightarrow \quad \sum_{i=k}^{\infty} f(i) < f(k) + \int_k^\infty f(x)\,dx$$

The right side can be re-arranged as

$$\sum_{i=k+1}^{\infty} f(i-1) = \sum_{i=k}^{\infty} f(i) > \int_k^\infty f(x)\,dx \quad \Rightarrow \quad \int_k^\infty f(x)\,dx < \sum_{i=k}^{\infty} f(i)$$

Also note that the $i^{\text{th}}$ term of the series is $a_i = f(i)$. It then follows that

$$\boxed{\int_k^\infty f(x)\,dx \le \sum_{n=k}^{\infty} a_n \le a_k + \int_k^\infty f(x)\,dx}$$

The convergence or divergence of the series is therefore linked completely to the convergence or divergence of the improper integral.

The double inequality also allows upper and lower bounds to be evaluated for the sum of a convergent series whose terms form a sequence drawn from a positive, continuous and decreasing function $f(x)$.

The **convergence of the *p*-series** can be established by a combination of the divergence and integral tests:

$$\lim_{n\to\infty} \frac{1}{n^p} = \begin{cases} \infty & (p<0) \\ 1 & (p=0) \\ 0 & (p>0) \end{cases} \qquad \Rightarrow \qquad \text{the } p\text{-series diverges for } p \le 0.$$

When $p > 0$ the function $f(x) = \dfrac{1}{x^p}$ is positive, continuous and decreasing on $[1, \infty)$ so that we may use the integral test.

$$\int_1^\infty x^{-p}\,dx = \left\{ \begin{array}{ll} \displaystyle\lim_{n\to\infty} \left[ \frac{x^{-p+1}}{-p+1} \right]_1^n & (p \ne 1) \\ \displaystyle\lim_{n\to\infty} \Big[ \ln x \Big]_1^n & (p=1) \end{array} \right\} = \begin{cases} \dfrac{1}{p-1} & (p>1) \\ \infty & (0<p\le 1) \end{cases}$$

Therefore the *p*-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges otherwise.

Another example of the integral test arises in showing that

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if and only if $p > 1$ [– details omitted].

END OF CHAPTER 6