7. Partial Differentiation

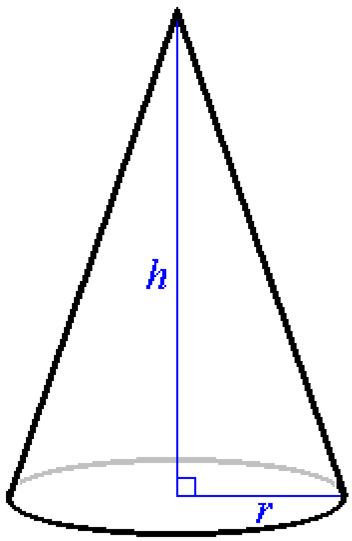
For functions of one variable, $y = f(x)$, the rate of change of the dependent variable can be found unambiguously by differentiation: $\frac{dy}{dx} = f'(x)$. In this chapter we explore rates of change for functions of more than one variable, such as $z = f(x, y)$.

Contents:

7.1 Partial Derivatives - introduction, chain rule, practice

Example 7.1.1

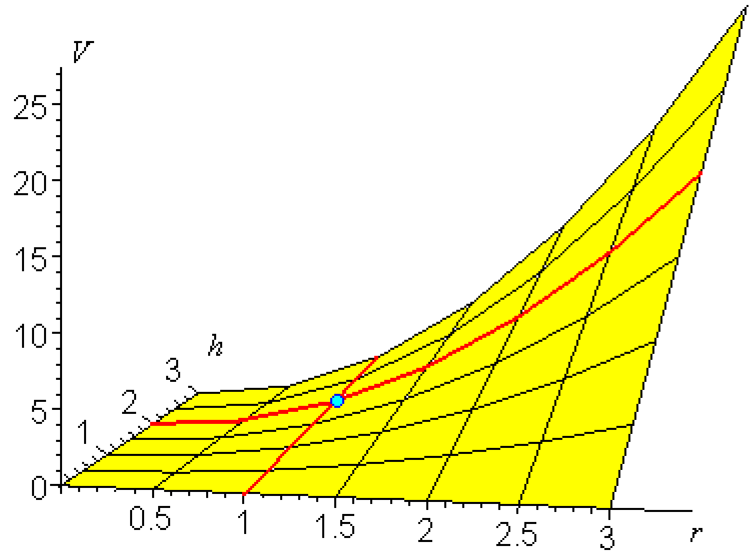
At a particular instant, a cone has a height of $h = 2$ m and a base radius of $r = 1$ m.
The base radius is increasing at a rate of 1 mm/s. The height is constant.
How fast is the volume V increasing at this time?



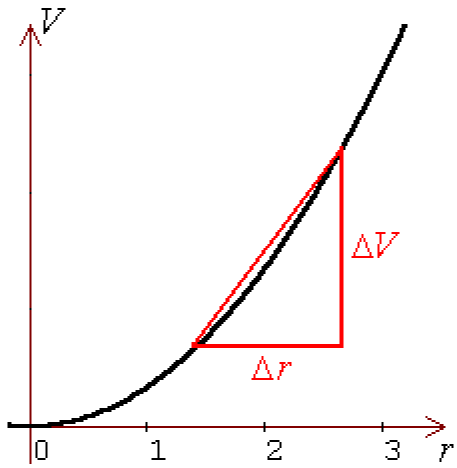
But if $\frac{dr}{dt} = 1\,\text{mm s}^{-1}$ and $\frac{dh}{dt} = -2\,\text{mm s}^{-1}$,
how do we find $\frac{dV}{dt}$?
We shall return to this question later.

Graph of V against r and h :

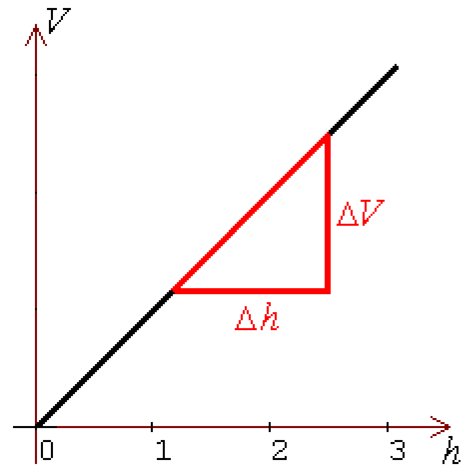
Plotting $z = V$ (where $V = \pi r^2 h / 3$) against both $x = r$ and $y = h$ yields



The cross-section of this surface in the vertical plane $h = 2$ is



The cross-section of this surface in the vertical plane $r = 1$ is



The tangent line to the surface in a cross-section (*h* = constant) has a slope of $\dfrac{\partial V}{\partial r}$.

The tangent line to the surface in a cross-section (*r* = constant) has a slope of $\dfrac{\partial V}{\partial h}$.

At each point on the surface, these two tangent lines define a tangent plane.

V is a function of *r* and *h*, each of which in turn is a function of *t* only.
In this case, the chain rule becomes

$$\boxed{\frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}}$$

In example 7.1.1,

A Maple worksheet, used to generate the graph of $V(r,h) = \frac{1}{3}\pi r^2 h$, is available at "`http://www.engr.mun.ca/~ggeorge/3425/demos/conevolume.mws`".

Open this worksheet in Maple and click on the graph.
Then, by dragging the mouse (with left button down), one can change the direction of view of the graph as one wishes. Other features of the graph may be changed upon opening a menu with a right mouse click on the graph or by using the main menu at the top of the Maple window.

<u>Example 7.1.2</u>

$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

Find $f_z(0, 3, 4)$

[the first partial derivative of f with respect to z, evaluated at the point (0, 3, 4)].

Example 7.1.3

$$u = x^{y/z}$$

Find the three first partial derivatives, u_x, u_y, u_z.

7.2 <u>Higher Partial Derivatives</u>, Clairaut's theorem, Laplace's PDE

$$u_x = \frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) =$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) =$$

$$\frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial y\,\partial x}\right) =$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial y\,\partial x}\right) =$$

etc.

<u>Example 7.2.1</u>

$$u = x e^{-t} \sin y$$

Find the second partial derivatives u_{xy}, u_{yx}, u_{xx} and the third partial derivative u_{ttt}.

Clairaut's Theorem

If, on a disk D containing the point (a, b), f is defined and both of the partial derivatives f_{xy} and f_{yx} are continuous, (which is the case for most functions of interest), then

$$\boxed{f_{xy}(a,b) = f_{yx}(a,b)}$$

that is, the order of differentiation doesn't matter.

One of the most important partial differential equations involving second partial derivatives is **Laplace's equation**, which arises naturally in many applications, including electrostatics, fluid flow and heat conduction:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

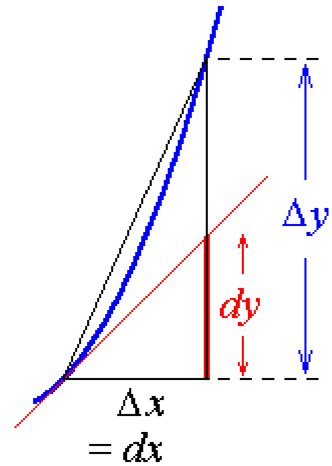
or its equivalent in $\mathbb{R}^3$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Example 7.2.2

Does $u = \ln\sqrt{x^2 + y^2}$ satisfy Laplace's equation?

7.3 Differentials; error estimation; chain rule [again]; implicit functions; partial derivatives on curves of intersection



In $\mathbb{R}^2$, let a curve have the Cartesian equation $y = f(x)$.

The small change in y, (Δy), caused by travelling along the curve for a small horizontal distance Δx, may be approximated by the change dy that is caused by travelling for the same horizontal distance Δx along the tangent line instead.

The exact form is $\Delta y = f(x+\Delta x) - f(x)$.

The approximation to Δy is

$$\Delta y \approx dy = f'(x)\,dx$$

where the increment Δx has been replaced by the differential dx.
The approximation improves as Δx decreases towards zero.

Stepping up one dimension, let a surface have the Cartesian equation $z = f(x, y)$.
The change in the dependent variable z caused by small changes in the independent variables x and y has the exact value

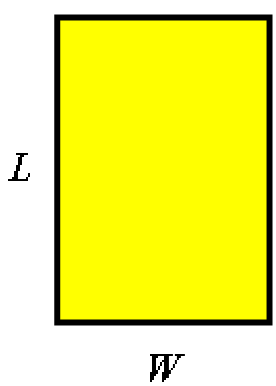
$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y).$$

The approximation to Δz is

$$\boxed{\Delta z \approx dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}$$

Example 7.3.1

A rectangle has quoted dimensions of 30 cm for length and 24 cm for width.
However, there may be an error of up to 1 mm in the measurement of each dimension.
Estimate the maximum error in the calculated area of the rectangle.



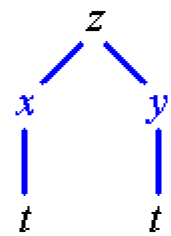
Let A = area, L = length and W = width.

Length = (30 ± 0.1) cm $\Rightarrow$ $L = 30$ and $\Delta L = 0.1$
Width = (24 ± 0.1) cm $\Rightarrow$ $W = 24$ and $\Delta W = 0.1$

Example 7.3.1 (continued)

Chain Rule

$z = f(x, y)$.
If x and y are both functions of t only, then, by the chain rule,

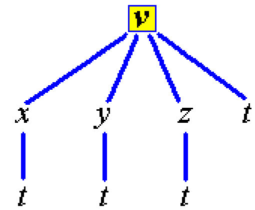


If $z = f(x, y)$ and y in turn is a function of x only, then replace t by x in the formula above.

Example 7.3.2

In the study of fluid dynamics, one approach is to follow the motion of a point in the fluid. In that approach, the velocity vector is a function of both time and position, while position, in turn, is a function of time. $\vec{\mathbf{v}} = \vec{\mathbf{v}}(\vec{\mathbf{r}}, t)$ and $\vec{\mathbf{r}} = \vec{\mathbf{r}}(t)$.

The acceleration vector is then obtained through differentiation following the motion of the fluid:



$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}}{dt} = \frac{\partial \vec{\mathbf{v}}}{\partial t} + \frac{\partial \vec{\mathbf{v}}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{\mathbf{v}}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{\mathbf{v}}}{\partial z}\frac{dz}{dt}$$

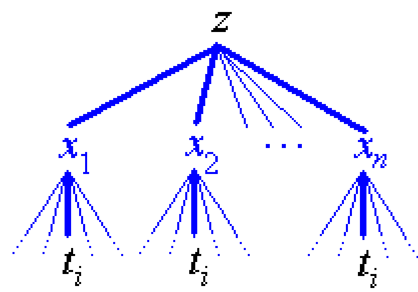
or, equivalently, $$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}}{dt} = \frac{\partial \vec{\mathbf{v}}}{\partial t} + \left(\vec{\mathbf{v}} \bullet \vec{\nabla}\right)\vec{\mathbf{v}}$$

Further analysis of an ideal fluid of density ρ at pressure p subjected to a force field **F** leads to Euler's equation of motion

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} + \left(\vec{\mathbf{v}} \bullet \vec{\nabla}\right)\vec{\mathbf{v}} = \vec{\mathbf{F}} - \frac{\vec{\nabla} p}{\rho}$$

This application of partial differentiation will be explored in some majors in a later semester. As a simple example here, suppose that $\vec{\mathbf{v}} = \left[\begin{array}{ccc} e^{-x} & 1 & 10(1-t) \end{array}\right]^{\mathrm{T}}$, then find the acceleration vector.

Generalized Chain Rule



Let z be a function of n variables $\{ x_1, x_2, \ldots, x_n \}$, each of which, in turn, is a function of m variables $\{ t_1, t_2, \ldots, t_m \}$, so that

$$z = f\left(x_1\left(t_1, t_2, \ldots, t_m\right), x_2\left(t_1, t_2, \ldots, t_m\right), \ldots \right.$$
$$\left. \ldots, x_n\left(t_1, t_2, \ldots, t_m\right)\right).$$

To find $\dfrac{\partial z}{\partial t_i}$,

trace all paths that start at z and end at t_i, via all of the $\{ x_j \}$ variables.

$$\boxed{dz = \sum_{j=1}^{n} \frac{\partial z}{\partial x_j} dx_j \quad \text{and} \quad \frac{\partial z}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial z}{\partial x_j} \frac{\partial x_j}{\partial t_i} \quad (i = 1, 2, \cdots, m)}$$

<u>Example 7.3.3</u>:

$u = xy + yz + zx$, $\quad x = st$, $\quad y = e^{st}$ and $z = t^2$.
Find u_s in terms of s and t only. Find the value of u_s when $s = 0$ and $t = 1$.

Implicit functions:

If z is defined implicitly as a function of x and y by $F(x, y, z) = c$, then
$dF = F_x\,dx + F_y\,dy + F_z\,dz = 0$
$$\Rightarrow \quad dz = -\frac{1}{F_z}\left(F_x\,dx + F_y\,dy\right) \quad \text{provided} \quad F_z = \frac{\partial F}{\partial z} \neq 0$$

Example 7.3.4:

Find the change in z when x and y both increase by 0.2 from the point (1, 2, 2) on the sphere $x^2 + y^2 + z^2 = 9$.

$$F = x^2 + y^2 + z^2 = 9 \quad \Rightarrow \quad dz =$$

Solution − Approximate motion on the sphere by motion on the tangent plane:

Curves of Intersection

Example 7.3.5:
Find both partial derivatives with respect to z on the curve of intersection of the sphere centre the origin, radius 5, and the circular cylinder, central axis on the y-axis, radius 3.

Sphere: $f =$

Cylinder: $g =$

$\Rightarrow$ $df =$

and $dg =$

which leads to the linear system

Example 7.3.6

A surface is defined by $f(x, y, z) = xz + y^2 z + z = 1$. Find $\dfrac{\partial z}{\partial y}$.

Implicit method:

$df =$

In general, if $a\,dx + b\,dy = c\,dz$ then
$\dfrac{\partial z}{\partial x} = \dfrac{a}{c}$ (because y is constant and $dy = 0$ in the slice in which $\dfrac{\partial z}{\partial x}$ is evaluated) and
$\dfrac{\partial z}{\partial y} = \dfrac{b}{c}$ (because x is constant and $dx = 0$ in the slice in which $\dfrac{\partial z}{\partial y}$ is evaluated).

7.4 <u>The Jacobian</u> - implicit and explicit forms; plane polar; spherical polar

The Jacobian is a conversion factor for a differential of area or volume between one orthogonal coordinate system and another.

Let (x, y), (u, v) be related by the pair of simultaneous equations

$$f(x, y, u, v) = c_1$$
$$g(x, y, u, v) = c_2$$

$\Rightarrow$ $df = f_x\,dx + f_y\,dy + f_u\,du + f_v\,dv = 0$

and $dg = g_x\,dx + g_y\,dy + g_u\,du + g_v\,dv = 0$

The Jacobian for the transformation from (x, y) to (u, v) is also the magnitude of the cross product of the tangent vectors that define the boundaries of the element of area, so that

$$\boxed{\frac{\partial(x, y)}{\partial(u, v)} = \left| \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v} \right|}$$

Example 7.4.1

Transform the element of area $dA = dx\,dy$ to plane polar coordinates.

$$x = r\cos\theta\,, \quad y = r\sin\theta\,.$$

If x, y can be written as explicit functions of (u, v), then an explicit form of the Jacobian is available:

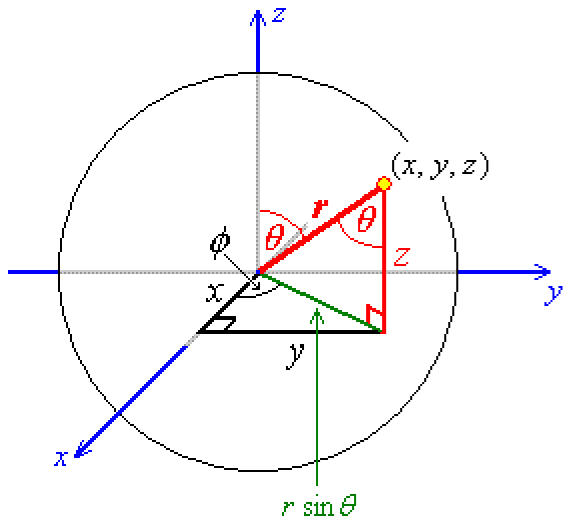
$$\boxed{\frac{\partial(x, y)}{\partial(u, v)} = \left\| \begin{matrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{matrix} \right\|}$$

The Jacobian can also be used to express an element of volume in terms of another orthogonal coordinate system:

$$\boxed{dV = dx\,dy\,dz = \frac{\partial(x, y, z)}{\partial(u, v, w)}\,du\,dv\,dw}$$

Spherical Polar Coordinates

The "declination" angle θ is the angle between the positive z axis and the radius vector $\vec{\mathbf{r}}$. $0 \le \theta \le \pi$.



The "azimuth" angle ϕ is the angle on the x-y plane, measured anticlockwise from the positive x axis, of the shadow of the radius vector. $0 \le \phi < 2\pi$.

$$z = r\cos\theta$$

The shadow of the radius vector on the x-y plane has length $r\sin\theta$.

It then follows that

$$x = r\sin\theta\cos\phi \quad \text{and} \quad y = r\sin\theta\sin\phi .$$

Example 7.4.2

Express the element of volume dV in spherical polar coordinates,

$$\vec{\mathbf{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{bmatrix}$$

Using the explicit form,

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \left\| \begin{matrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{matrix} \right\| =$$

For a transformation from Cartesian to plane polar coordinates in $\mathbb{R}^2$,

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r \quad \text{so that} \quad dA = dx\, dy = r\, dr\, d\theta$$

For cylindrical polar coordinates in $\mathbb{R}^3$,

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r \quad \Rightarrow \quad dV = r\, dr\, d\theta\, dz$$

Explicit method for plane polar coordinates:

$x = r\cos\theta\,, \quad y = r\sin\theta$

$$\Rightarrow \quad \frac{\partial(x, y)}{\partial(r, \theta)} = ABS\left(\det\begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix}\right) = \left\| \begin{matrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{matrix} \right\|$$

$$= r\cos^2\theta + r\sin^2\theta = r\,.$$

Implicit method for plane polar coordinates [Example 7.4.1]:

$$f = x - r\cos\theta = 0 \quad \Rightarrow \quad df = dx - \cos\theta\, dr + r\sin\theta\, d\theta = 0$$

$$g = y - r\sin\theta = 0 \quad \Rightarrow \quad dg = dy - \sin\theta\, dr - r\cos\theta\, d\theta = 0$$

$$\Rightarrow \quad \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{A}} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}}_{\mathbf{B}} \cdot \begin{bmatrix} dr \\ d\theta \end{bmatrix}$$

$$\Rightarrow \quad \frac{\partial(x, y)}{\partial(r, \theta)} = \left| \frac{|\mathbf{B}|}{|\mathbf{A}|} \right| = \left| \frac{r\cos^2\theta + r\sin^2\theta}{1} \right| = |r| = r$$

Note that for spherical polar coordinates (r, θ, ϕ) there is no agreement among textbooks as to which angle is θ and which angle is ϕ. In this course we adopt θ as the angle between the positive z axis and the radius vector $\vec{\mathbf{r}}$. You are likely to encounter textbooks and software in which that same angle is labelled as ϕ.

7.5 <u>Gradient Vector</u>, directional derivative, potential function, central force law

Let $F = F(\vec{\mathbf{r}})$ and $\vec{\mathbf{r}} = \vec{\mathbf{r}}(t)$.

$$\frac{dF}{dt} =$$

Directional Derivative

The rate of change of the function F at the point P_o in the direction of the vector $\vec{\mathbf{a}} = a\hat{\mathbf{a}}$ $(a \neq 0)$ is

$$\boxed{D_{\hat{\mathbf{a}}} F\big|_{P_o} = \vec{\nabla} F\big|_{P_o} \bullet \hat{\mathbf{a}}}$$

The maximum value of the directional derivative of F at any point P_o occurs when

Example 7.5.1

The temperature in a region within 10 units of the origin follows the form
$T = r\,e^{-r}$, where $r = \sqrt{x^2 + y^2 + z^2}$.
Find the rate of temperature change at the point (−1, −1, +1) in the direction of the vector
$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$.

Central Force Law

If $\phi(\vec{\mathbf{r}})$ is the **potential function** for some force per unit mass or force per unit charge $\vec{\mathbf{F}}(\vec{\mathbf{r}})$, then $\vec{\mathbf{F}} = -\vec{\nabla}\phi$ (or, in some cases, $\vec{\mathbf{F}} = \vec{\nabla}\phi$). For a central force law, the potential function is spherically symmetric and is dependent only on the distance r from the origin. When the central force law is a simple power law, $\phi = k\,r^n$.

Examples include the inverse square laws, for which $n = -1$:

Electromagnetism:

$$k = \frac{Q}{4\pi\varepsilon}, \quad \phi = \frac{Q}{4\pi\varepsilon\, r}, \quad \vec{\mathbf{F}} = -\vec{\nabla}\phi = \frac{Q}{4\pi\varepsilon\, r^2}\,\hat{\mathbf{r}}$$

Gravity:

$$k = -GM, \quad \phi = \frac{-GM}{r}, \quad \vec{\mathbf{F}} = -\vec{\nabla}\phi = \frac{-GM}{r^2}\,\hat{\mathbf{r}}$$

Some Applications of the Gradient Vector:

(1) The directional derivative D at the point $P_o(x_o, y_o, z_o)$ is maximized by choosing $\vec{\mathbf{a}}$ to be parallel to $\vec{\nabla} F$ at P_o, so that $D_{\hat{\mathbf{a}}} F \big|_{P_o} = \left| \vec{\nabla} F(\vec{\mathbf{r}}) \right|_{P_o}$.

(2) A normal vector to the surface $F(x, y, z) = c$ at the point $P_o(x_o, y_o, z_o)$ is $\vec{\mathbf{N}} = \vec{\nabla} F(\vec{\mathbf{r}}) \big|_{P_o}$.

(3) The equation of the line normal to the surface $F(x, y, z) = c$ at the point $P_o(x_o, y_o, z_o)$ is $\vec{\mathbf{r}} = \overrightarrow{OP_o} + t\left(\vec{\nabla} F \big|_{P_o}\right) = \vec{\mathbf{r}}_o + t\left(\vec{\nabla} F \big|_{P_o}\right)$

(4) The equation of the tangent plane to the surface $F(x, y, z) = c$ at the point $P_o(x_o, y_o, z_o)$ is $\left(\vec{\mathbf{r}} - \overrightarrow{OP_o}\right) \bullet \left(\vec{\nabla} F \big|_{P_o}\right) = \left(\vec{\mathbf{r}} - \vec{\mathbf{r}}_o\right) \bullet \left(\vec{\nabla} F \big|_{P_o}\right) = 0$

(5) If the point $P_o(x_o, y_o, z_o)$ lies on both of the surfaces $F(x, y, z) = c$ and $G(x, y, z) = k$, then the acute angle of intersection θ of the surfaces at the point is given by

$$\cos\theta = \frac{\left| \left(\vec{\nabla} F \big|_{P_o}\right) \bullet \left(\vec{\nabla} G \big|_{P_o}\right) \right|}{\left| \vec{\nabla} F \right|_{P_o} \cdot \left| \vec{\nabla} G \right|_{P_o}}$$

7.6 Extrema; Second Derivative Test for $z = f(x, y)$

Much of differential calculus in the study of maximum and minimum values of a function of one variable carries over to the case of a function of two (or more) variables. In order to visualize what is happening, we shall restrict our attention to the case of functions of two variables, $z = f(x, y)$.

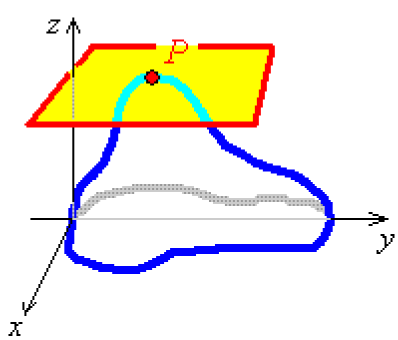
For a function $f(x, y)$ defined on some domain D in $\mathbb{R}^2$, the point $P(x_o, y_o)$ is a **critical point** [and the value $f(x_o, y_o)$ is a **critical value**] of f if

1) P is on any boundary of D; or
2) $f(x_o, y_o)$ is undefined; or
3) f_x and/or f_y is undefined at P; or
4) f_x and f_y are both zero at P ($\Rightarrow$ $\vec{\nabla} f = \vec{\mathbf{0}}$ at P).

A local maximum continues to be equivalent to a "hilltop",

while a local minimum continues to be equivalent to a "valley bottom".

"Local extremum" is a collective term for local maximum or local minimum.



Instead of tangent lines being horizontal at critical points of type (4), we now have tangent *planes* being horizontal at critical points of type (4).

At any local extremum at which $f(x, y)$ is differentiable, the tangent plane is horizontal, $f_x = f_y = 0$ and $\vec{\nabla} f = \vec{\mathbf{0}}$.

The converse is false.

$\vec{\nabla} f = \vec{\mathbf{0}}$ does *not* guarantee a local extremum. There could be a saddle point (the higher dimensional equivalent of a point of inflection) instead.

Example 7.6.1

Find and identify the nature of the extrema of $f(x, y) = x^2 + y^2 + 4x - 6y$.

Polynomial functions of x and y are defined and are infinitely differentiable on *all* of $\mathbb{R}^2$. Therefore the only critical points are of type (4).

Second Derivative Test

To determine the nature of a critical point:

1) Examine the values of f in the neighbourhood of P; ***or***
2) [First derivative test:] Examine the changes in f_x and f_y at P; ***or***
3) Use the second derivative test:

At all points (a, b) where $\vec{\nabla} f = \vec{\mathbf{0}}$, find all second partial derivatives, then find

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \qquad \text{(the “Hessian” determinant)}$$

and evaluate D at (x, y) = (a, b).

$D(a,b) > 0$ and $f_{xx}(a,b) > 0 \Rightarrow$ a relative **minimum** of f is at (a, b)

$D(a,b) > 0$ and $f_{xx}(a,b) < 0 \Rightarrow$ a relative **maximum** of f is at (a, b)

$D(a,b) < 0 \Rightarrow$ a **saddle point** of f is at (a, b)

$D(a,b) = 0 \Rightarrow$ test fails (no information).

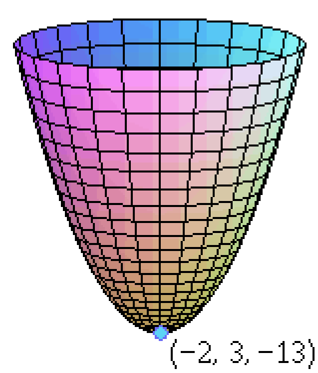
Note that $D(a,b) > 0 \quad \Rightarrow \quad f_{xx} f_{yy} > f_{xy} f_{yx}$ and $f_{xy} f_{yx} = \left(f_{xy}\right)^2 \geq 0$

Therefore $D > 0 \quad \Rightarrow \quad f_{xx} f_{yy} > 0 \quad \Rightarrow \quad f_{xx},\, f_{yy}$ must have the same sign at an extremum.

One can therefore check the sign of whichever of f_{xx} or f_{yy} is easier to evaluate in the second derivative test, when determining whether an extremum is a maximum or a minimum.

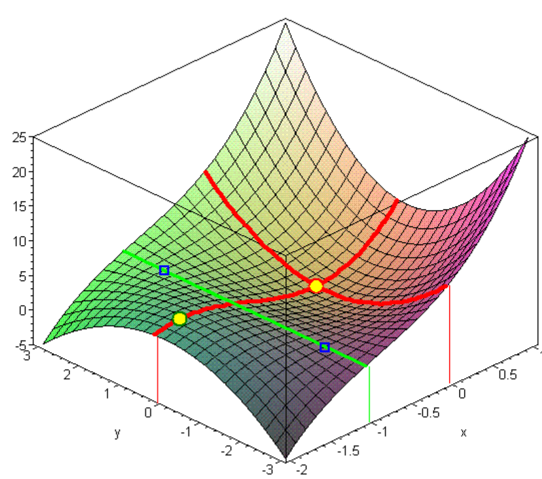
Example 7.6.1 (again):

Find all extrema of $f(x,y) = x^2 + y^2 + 4x - 6y$.



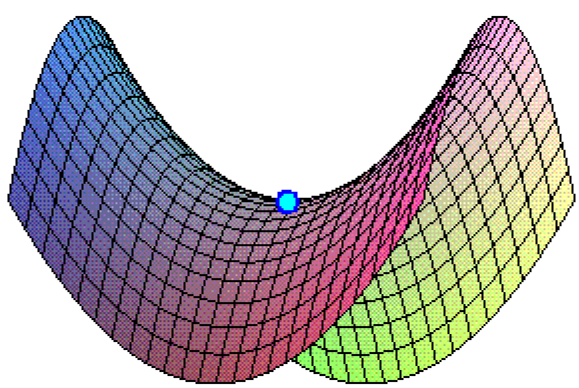
Example 7.6.2
Find all extrema of $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$.

Example 7.6.2 (continued)



Example 7.6.3

Find all extrema of $z = f(x, y) = x^2 - y^2$



Example 7.6.4

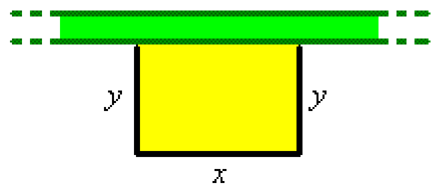
Find and identify the nature of the extrema of $f(x, y) = \sqrt{1 - x^2 - y^2}$.

7.7 Lagrange Multipliers; nearest point on curve of intersection to given point

In order to obtain an appreciation for the geometric foundation for the method of Lagrange multipliers, we shall begin with an example that could be solved in another way.

Example 7.7.1

A farmer wishes to enclose a rectangle of land. One side is a straight hedge, more than 30 m long. The farmer has a total length of 12 m of fencing available for the other three sides. What is the greatest area that can be enclosed by the available fencing?



The function to be maximized is the area enclosed by the fencing and the hedge:

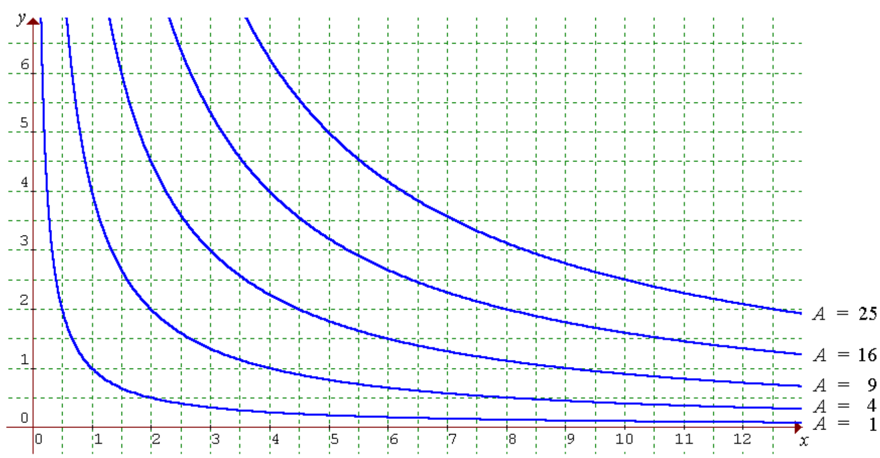
$$A(x, y) = x\,y$$

The constraint is the length of fencing available:

$$L(x, y) = x + 2y = 12$$

There are additional constraints. Neither length may be a negative number.
Therefore any solution is confined to the first quadrant of the (x, y) graph.

Superimpose the graph of the constraint function $L(x, y) = 12$ on the contour graph of the function $z = A(x, y)$:



The least value of $A(x, y)$ is zero, on the coordinate axes.
At $(0, 6)$, two 6-metre segments of fence are touching, enclosing zero area.
At $(12, 0)$, all of the fence is up against the hedge, enclosing zero area.

Example 7.7.1 (continued)

As one travels along the constraint line $L = 12$ from one absolute minimum at (0, 6) to the other absolute minimum at (12, 0), the line first passes through increasing contours of $A(x, y)$,

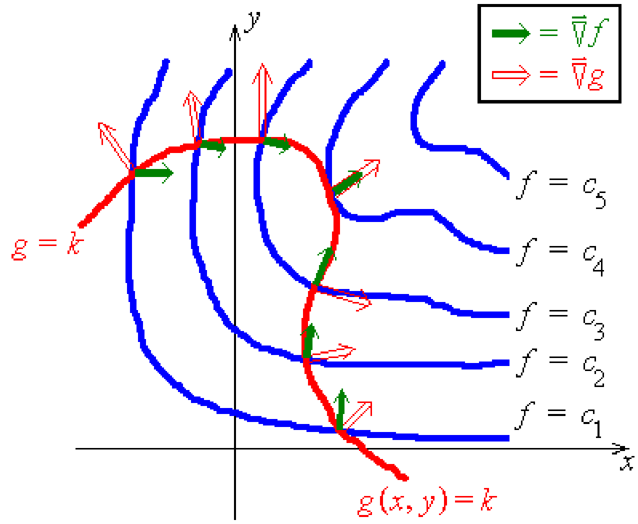
$$A\left(6-\sqrt{34}, 3+\sqrt{8.5}\right) = 1$$
$$A\left(6-2\sqrt{7}, 3+\sqrt{7}\right) = 4$$
$$A\left(6-\sqrt{18}, 3+\sqrt{4.5}\right) = 9$$
$$A(4, 4) = 16$$

until, somewhere in the vicinity of (6, 3), the area reaches a maximum value, then declines, (for example, $A(8, 2) = 16$), back to the other absolute minimum at (12, 0).

At the maximum, the graph of $L(x, y) = 12$

λ is the Lagrange multiplier.

To maximize $f(x, y)$ subject to the constraint $g(x, y) = k$:



As one travels along the constraint curve $g(x, y) = k$, the maximum value of $f(x, y)$ occurs only where the two gradient vectors are parallel. This principle can be extended to the case of functions of more than two variables.

<u>General Method of Lagrange Multipliers</u>

To find the maximum or minimum value(s) of a function $f(x_1, x_2, \ldots, x_n)$ subject to a constraint $g(x_1, x_2, \ldots, x_n) = k$, solve the system of simultaneous (usually non-linear) equations in $(n + 1)$ unknowns:

$$\begin{aligned} \vec{\nabla} f &= \lambda \vec{\nabla} g \\ g &= k \end{aligned}$$

where λ is the Lagrange multiplier.
Then identify which solution(s) gives a maximum or minimum value for f.

This can also be extended to the case of more than one constraint:

In the presence of two constraints $g(x_1, x_2, \ldots, x_n) = k$ and $h(x_1, x_2, \ldots, x_n) = c$, solve the system in $(n + 2)$ unknowns:

$$\begin{aligned} \vec{\nabla} f &= \lambda \vec{\nabla} g + \mu \vec{\nabla} h \\ g &= k \\ h &= c \end{aligned}$$

<u>Example 7.7.2</u>

Find the points, on the curve of intersection of the surfaces $x + y - 2z = 6$ and $2x^2 + 2y^2 = z^2$, that are nearest / farthest from the origin.

The maximum and minimum values of distance d occur at the same place as the maximum and minimum values of d^2. [Differentiation of d^2 will be much easier than differentiation of d.]

The function to be maximized/minimized is

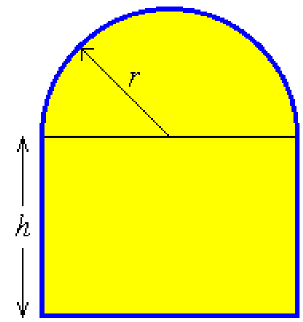
The constraints are

Example 7.7.2 (continued)

7.8 Miscellaneous Additional Examples

Example 7.8.1

A window has the shape of a rectangle surmounted by a semicircle. Find the maximum area of the window, when the perimeter is constrained to be 8 m.



The function to be maximized is

The constraint is the perimeter function

Single variable method:

Example 7.8.1 (continued)

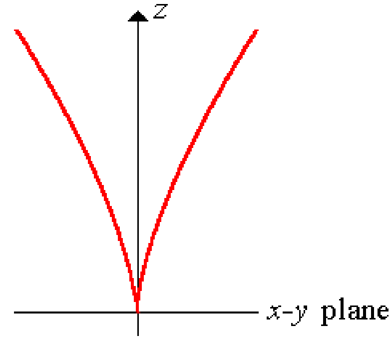
Lagrange Multiplier Method:

Example 7.8.1 (continued)

Example 7.8.2

Find the local and absolute extrema of the function

$$f(x, y) = \sqrt[3]{x^2 + y^2} .$$



[Any vertical cross-section containing the z axis.]

Example 7.8.3

Find the maximum and minimum values of the function

$$V(x, y) = 48xy - 32x^3 - 24y^2$$

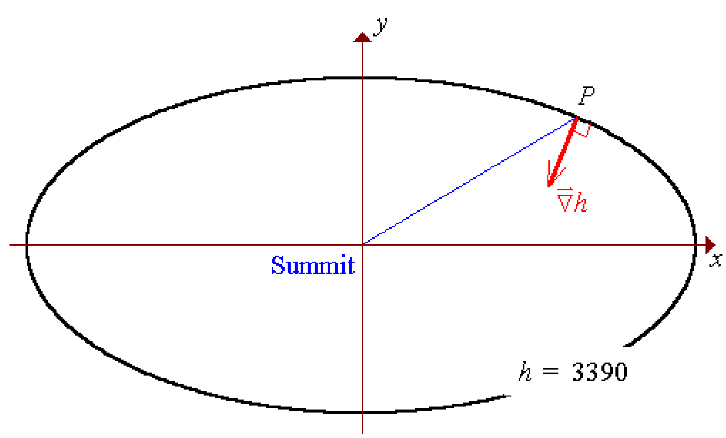
in the unit square $0 \le x \le 1$, $0 \le y \le 1$.

Example 7.8.4

A hilltop is modelled by the part of the elliptic paraboloid

$$h(x,y) = 4000 - \frac{x^2}{1000} - \frac{y^2}{250}$$

that is above the *x*-*y* plane. At the point *P*(500, 300, 3390), in which direction is the steepest ascent?



Example 7.8.5

Show that $u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right)$ satisfies $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$.

[Space for additional notes]

[END OF CHAPTER 7]