8. Multiple Integration

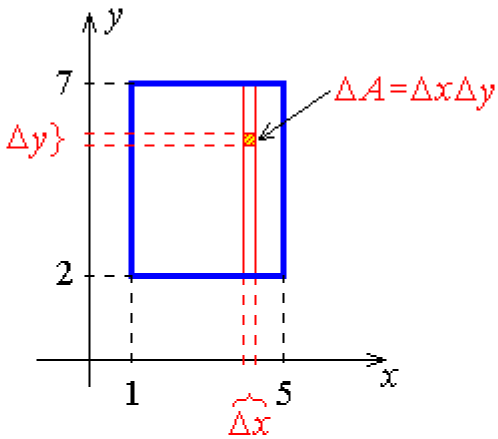
This chapter provides only a very brief introduction to the major topic of multiple integration. Uses of multiple integration include the evaluation of areas, volumes, masses, total charge on a surface and the location of a centre-of-mass. The issue of integration over non-flat surfaces is beyond the scope of ENGI 3425.

Contents:

8.1 Double Integrals (Cartesian Coordinates)

Example 8.1.1

Find the area shown (assuming SI units).



Example 8.1.1 (continued)

Suppose that the surface density on the rectangle is $\sigma = x^2 y$. Find the mass of the rectangle.

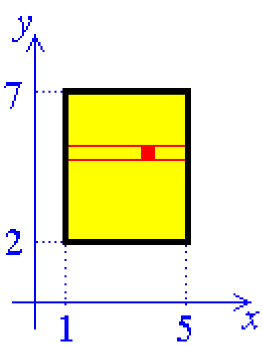
The element of mass is

OR

We can choose to sum horizontally first:

$$m = \int_2^7 \int_1^5 x^2 y \, dx \, dy$$

$$m = \int_2^7 y \left(\int_1^5 x^2 \, dx \right) dy$$



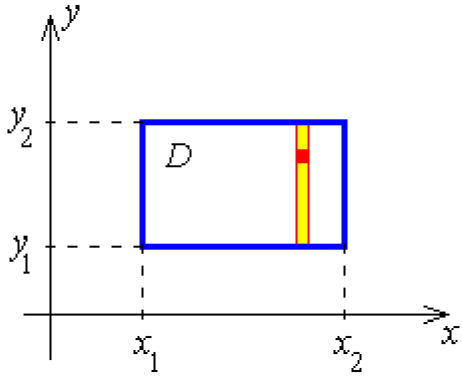
The inner integral has no dependency at all on y, in its limits or in its integrand. It can therefore be extracted as a "constant" factor from inside the outer integral.

$$m = \left(\int_1^5 x^2 \, dx \right) \left(\int_2^7 y \, dy \right)$$

which is exactly the same form as before, leading to the same value of 930 kg.

A double integral $\iint_D f(x, y)\, dA$ may be separated into a pair of single integrals if

- the region D is a rectangle, with sides parallel to the coordinate axes; and
- the integrand is separable: $f(x, y) = g(x)\, h(y)$.



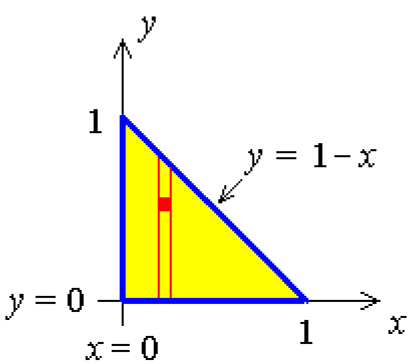
$$\iint_D f(x, y)\, dA = \int_{x_1}^{x_2} \int_{y_1}^{y_2} g(x) h(y)\, dy\, dx$$

$$= \left(\int_{x_1}^{x_2} g(x)\, dx \right) \left(\int_{y_1}^{y_2} h(y)\, dy \right)$$

This was the case in Example 8.1.1.

Example 8.1.2

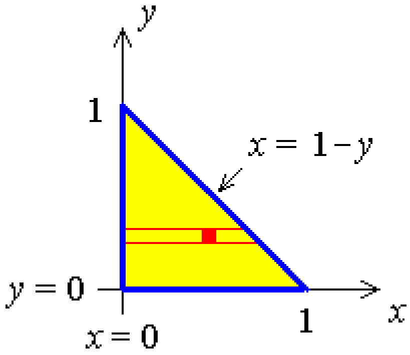
The triangular region shown here has surface density $\sigma = x + y$.
Find the mass of the triangular plate.



Example 8.1.2 (continued)

OR

We can choose to sum horizontally first (re-iterate):



Generally:

In Cartesian coordinates on the *xy*-plane, the rectangular element of area is

$$\Delta A = \Delta x \, \Delta y \, .$$

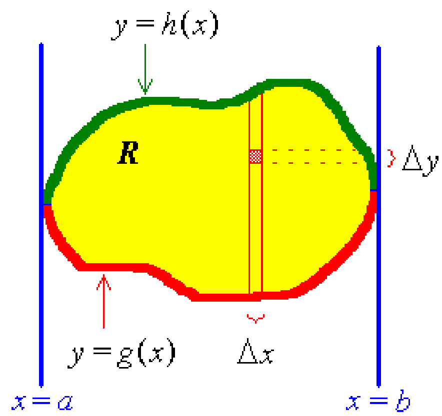
Summing all such elements of area along a vertical strip, the area of the elementary strip is

$$\left(\sum_{y=g(x)}^{h(x)} \Delta y \right) \Delta x$$

Summing all the strips across the region *R*, the total area of the region is:

$$A \approx \sum_{x=a}^{b} \left(\left(\sum_{y=g(x)}^{h(x)} \Delta y \right) \Delta x \right)$$

In the limit as the elements Δx and Δy shrink to zero, this sum becomes



$$A = \int_{x=a}^{b} \int_{y=g(x)}^{h(x)} 1 \, dy \, dx$$

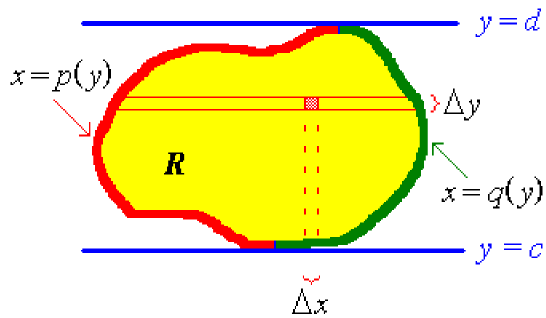
If the surface density σ within the region is a function of location, $\sigma = f(x, y)$, then the mass of the region is

$$m = \int_{x=a}^{b} \left(\int_{y=g(x)}^{h(x)} f(x, y) \, dy \right) dx$$

The inner integral must be evaluated first.

Re-iteration:

One may reverse the order of integration by summing the elements of area ΔA horizontally first, then adding the strips across the region from bottom to top. This generates the double integral for the total area of the region



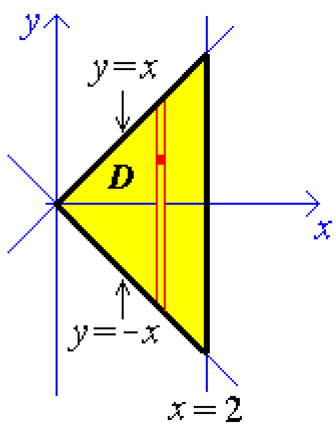
$$A = \int_{y=c}^{d} \left(\int_{x=p(y)}^{q(y)} 1 \, dx \right) dy$$

The mass becomes

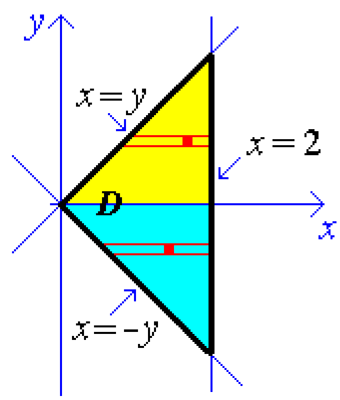
$$m = \int_{y=c}^{d} \left(\int_{x=p(y)}^{q(y)} f(x, y) \, dx \right) dy$$

Choose the orientation of elementary strips that generates the simpler double integration.

For example,



is preferable to



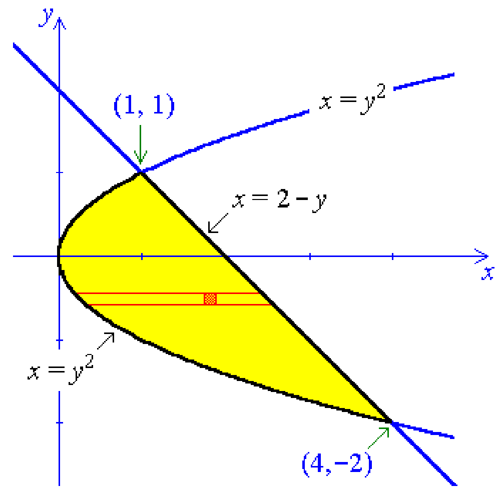
Example 8.1.3

Evaluate $I = \iint_R \left(6x + 2y^2\right) dA$

where R is the region enclosed by the parabola $x = y^2$ and the line $x + y = 2$.

The upper boundary changes form at $x = 1$.
The left boundary is the same throughout R.
The right boundary is the same throughout R.
Therefore choose horizontal strips.

$$I = \int_{-2}^{1} \int_{y^2}^{2-y} \left(6x + 2y^2\right) dx\, dy$$



8.2 Double Integrals (Plane Polar Coordinates)

The Jacobian of the transformation from Cartesian to plane polar coordinates is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \left\| \begin{vmatrix} x_r & y_r \\ x_\theta & y_\theta \end{vmatrix} \right\| = r$$

The element of area is therefore

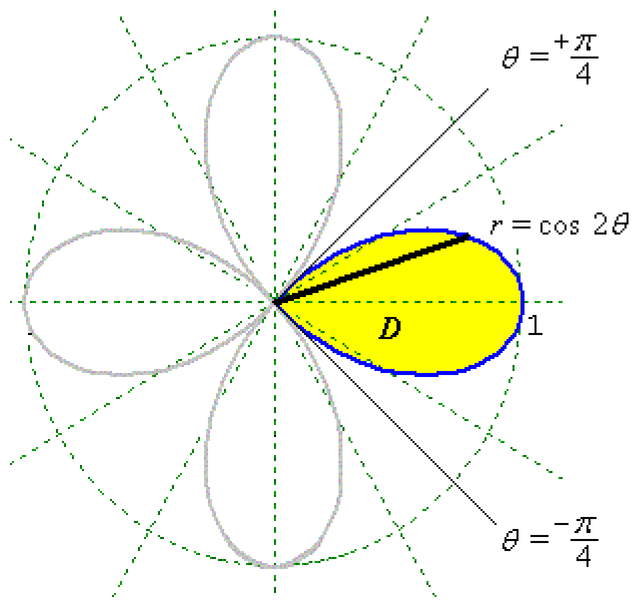
$$dA = dx\,dy = r\,dr\,d\theta$$

Example 8.2.1

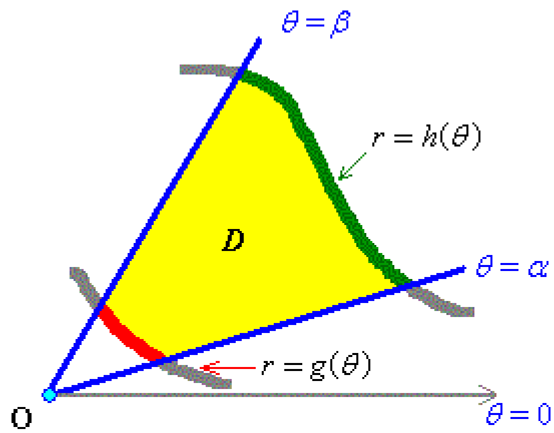
Find the area enclosed by one loop of the curve $r = \cos 2\theta$.

Boundaries:

Area:



In general, in plane polar coordinates,

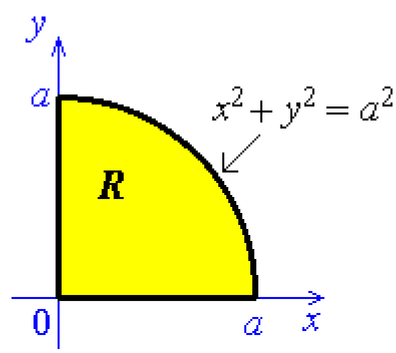


$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Example 8.2.2

Find the centre of mass for a plate of surface density $\sigma = \dfrac{k}{\sqrt{x^2+y^2}}$, whose boundary is the portion of the circle $x^2 + y^2 = a^2$ that is inside the first quadrant. k and a are positive constants.

Use plane polar coordinates.
Boundaries:
The positive x-axis is the line $\theta = 0$.
The positive y-axis is the line $\theta = \pi/2$.
The circle is $r^2 = a^2$, which is $r = a$.



Mass:

Surface density $\sigma = \dfrac{k}{\sqrt{x^2+y^2}}$.

Example 8.2.2 (continued)

First Moments about the *x*-axis:

8.3 Triple Integrals

The concepts for double integrals (surfaces) extend naturally to triple integrals (volumes). The element of volume, in terms of the Cartesian coordinate system (x, y, z) and another orthogonal coordinate system (u, v, w), is

$$dV = dx\,dy\,dz = \frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw$$

and

$$\iiint_V f(x,y,z)\,dV = \int_{w_1}^{w_2}\int_{v_1(w)}^{v_2(w)}\int_{u_1(v,w)}^{u_2(v,w)} f\big(x(u,v,w),y(u,v,w),z(u,v,w)\big)\frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw$$

The most common choices for non-Cartesian coordinate systems in $\mathbb{R}^3$ are:

Cylindrical Polar Coordinates:

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

for which the differential volume is

$$dV = \frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}\,d\rho\,d\phi\,dz = \rho\,d\rho\,d\phi\,dz$$

Spherical Polar Coordinates:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

for which the differential volume is

$$dV = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\,dr\,d\theta\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$

<u>Example 8.3.1</u>:

Verify the formula $V = \frac{4}{3}\pi a^3$ for the volume of a sphere of radius a.

Start by placing the origin at the centre of the sphere.

Example 8.3.2:

The density of an object is equal to the reciprocal of the distance from the origin.
Find the mass and the average density inside the sphere $r = a$.

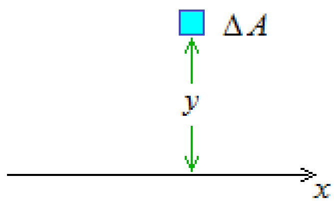
Use spherical polar coordinates.
Density:

Mass:

8.4 Second Moments of Area

The second moment, (also known as the second moment of area or the area moment of inertia) is a property of a cross section that can be used to predict the resistance of beams to bending and deflection, among other uses.

Just as the element of first moment of area about the *x*-axis is $\Delta M_x = y\,\Delta A$,



the element of second moment of area about the *x*-axis is $\Delta I_x = y^2 \Delta A$ and
the element of second moment of area about the *y*-axis is $\Delta I_y = x^2 \Delta A$.

Summing all such elements over a plane region *R* in the limit as $\Delta A \to 0$, for a region *R* of constant surface density, the second moments of area about the coordinate axes are

$$I_x = \iint_R y^2 dA \quad \text{and} \quad I_y = \iint_R x^2 dA$$

We are usually interested in the second moments about the centroid $(\bar{x}, \bar{y})$ (which is also the centre of mass when the density is constant):

$$I_x = \iint_R (y - \bar{y})^2 dA \quad \text{and} \quad I_y = \iint_R (x - \bar{x})^2 dA$$

Where possible, we choose coordinate axes that pass through the centroid.

We can also define the polar moment of area

$$I = I_x + I_y = \iint_R \left(x^2 + y^2\right) dA = \iint_R r^2 dA$$

A related concept is [mass] moment of inertia $I = \iint_R \left(x^2 + y^2\right) dm = \iint_R \left(x^2 + y^2\right) \sigma\, dA$,

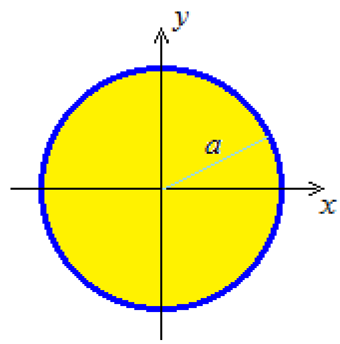
where $\sigma =$ surface density. The SI units of moment of inertia are kg m^2.

The kinetic energy of a rigid body rotating at angular speed ω about the origin is $E = \frac{1}{2} I \omega^2$. We shall not examine this application of second moments in ENGI 3425.

<u>Example 8.4.1</u>

Find the second moments of area for a uniform circular disc of radius *a*.

The centroid of the disc is at the centre of the circle.
Place the origin there. Use plane polar coordinates.



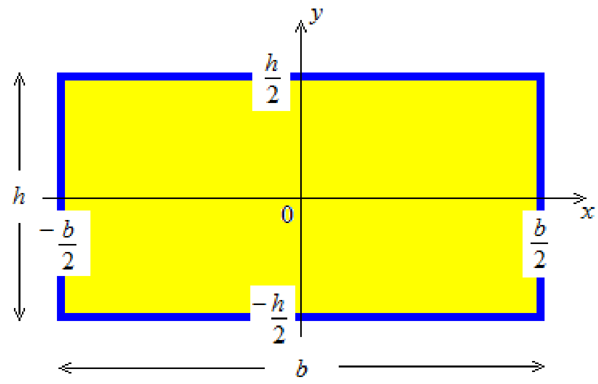
$$I_x = \iint_R y^2 \, dA =$$

<u>Example 8.4.2</u>

Find the second moments of area for a uniform rectangle of base b and height h.

Place the origin at the centre of the rectangle.

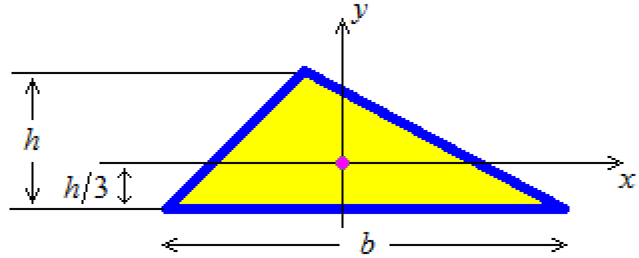
Use Cartesian coordinates.
Note that $A = bh$



$$I_x = \iint_R y^2 \, dA$$

Two other standard second moments of area about the centroid are:

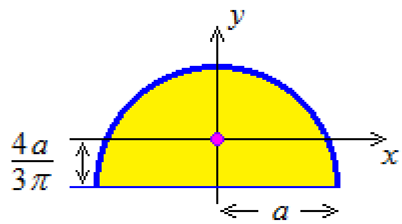
Triangle



$$A = \frac{bh}{2}$$

$$I_x = \frac{bh^3}{36}$$

Semi-circle



$$A = \frac{1}{2}\pi a^2$$

$$I_x = \frac{\pi a^4}{8}\left(1 - \frac{64}{9\pi^2}\right) \approx .28 I_y \quad \text{and} \quad I_y = \frac{\pi a^4}{8}$$

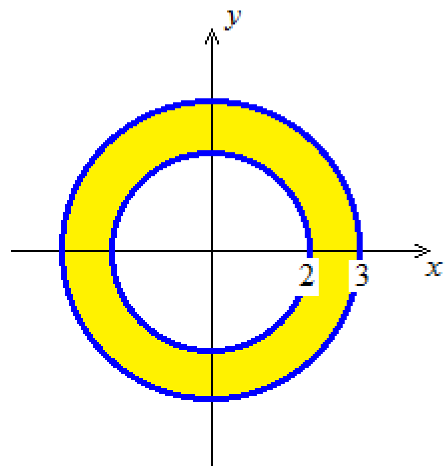
The second moment of area of a collection of regions that share the same centroid is just the sum of the separate second moments.

When a region has a hole in it, centered on the centroid of the complete region, then the second moment of area of the region is the difference between the second moment for the complete region (with the hole filled in) and the second moment for the hole.

Example 8.4.3

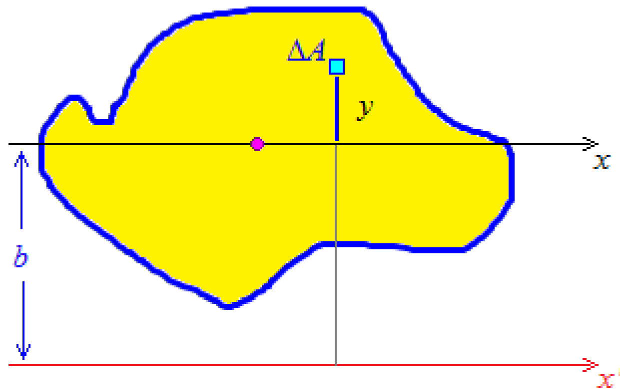
Find the second moments of area of an annulus (ring) of inner radius 2 cm and outer radius 3 cm about its centroid.

The region is the difference between two circular discs, which share the same centroid, at the origin.



Parallel Axis Theorem

The second moment of a composite shape can be found by shifting the reference axis of the standard second moment of each section from the centroidal axis of that section to a parallel axis that passes through the centroid of the composite shape.



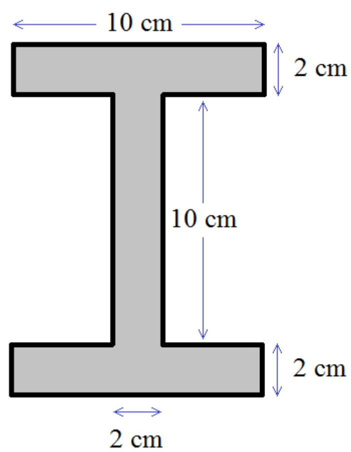
If the x axis passes through the centroid then the second moment $I_{x'}$ about an axis x' parallel to the x axis and a distance b away from it will be related to I_x:

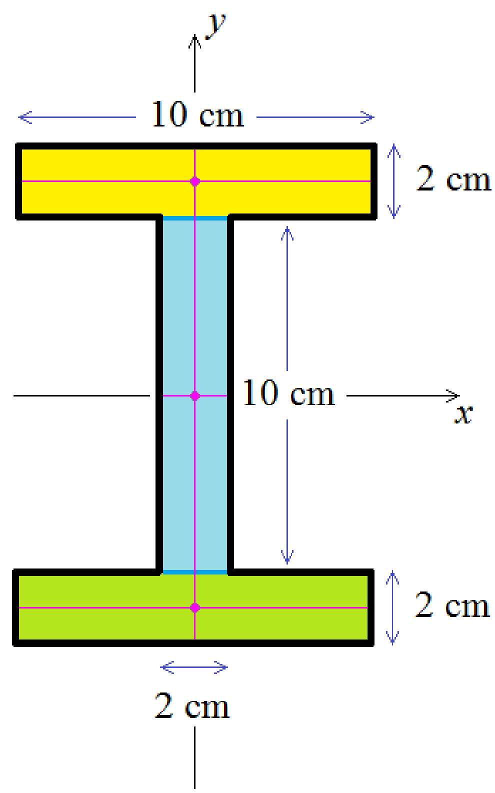
$$\Delta I_x = y^2 \Delta A \quad \Rightarrow \quad I_x = \iint_R y^2 \, dA$$

$$\Delta I_{x'} =$$

Example 8.4.4

Find the second moment of area of this cross section of a guide rail about its centroid.





Example 8.4.4 (continued)

Example 8.5.1

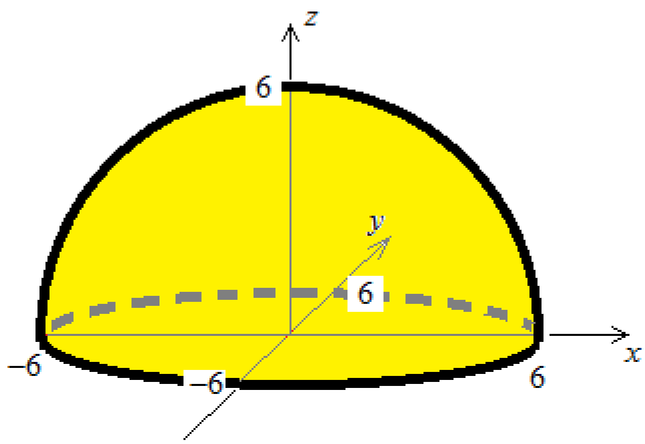
A swimming pool is filled to a depth of 2 m. It has a rectangular end wall of width 5 m. Find the force due to the water on the end wall. Assume that the density of the water is $\rho = 1000 \text{ kg m}^{-3}$ and that the acceleration due to gravity is $g = 9.81 \text{ m s}^{-2}$.

Example 8.5.2

A full trough of liquid of constant density ρ has a vertical side wall in the shape of a triangle joining the points $(-4, 4)$, $(2, 4)$ and the origin. Find the total hydrostatic force on this side wall, in terms of ρ and g.

Example 8.5.3

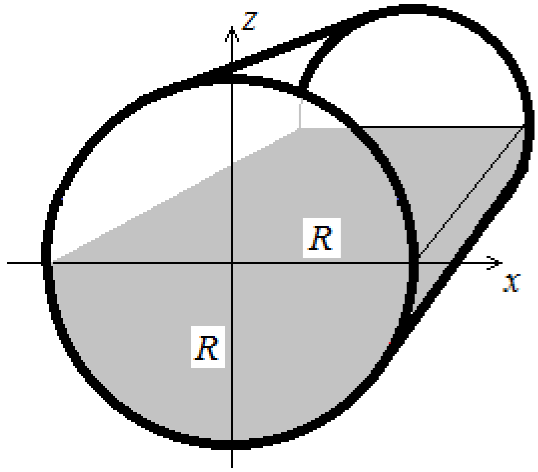
A hemisphere of radius 6 m has its centre at the origin, with its flat face on the equatorial plane (the *x*-*y* plane), such that $z \geq 0$ everywhere on the hemisphere. The interior of the sphere consists of material whose density is proportional to $\cos\theta$. Find the location of the centre of mass of the hemisphere.



Example 8.5.3 (continued)

<u>Example 8.5.4</u> (based on a question in the final examination of 2014)

A tank in the shape of a right circular cylinder of cross sectional radius R is lying on its curved side and is filled up to the half-way point with incompressible fluid of density ρ. Find the hydrostatic force on the semi-circular end wall due to the fluid (as a function of ρ, g and R).

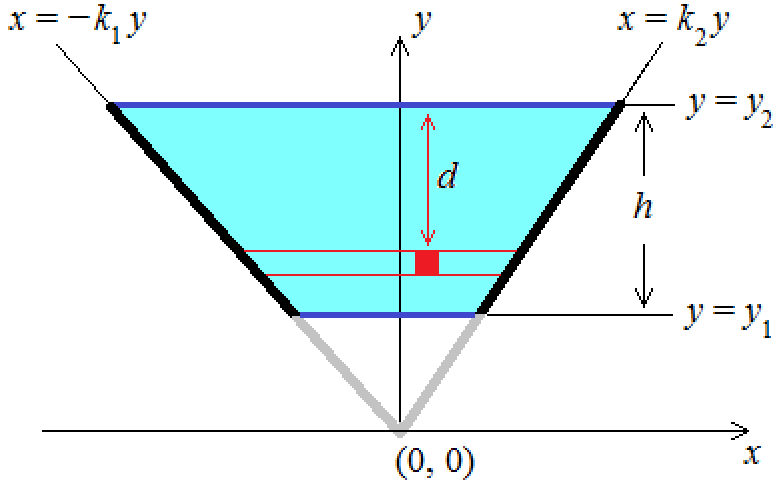


Method using double integration in plane polar coordinates:

Example 8.5.4 (continued)

Example 8.5.5

A trough has vertical trapezoidal end walls as shown in the diagram.



Find the total hydrostatic force F on this end wall due to a liquid of density ρ that fills it from its base at $y = y_1$ to its top at $y = y_2 = y_1 + h$.

Example 8.5.5 (continued)

Example 8.5.5 (continued)

Two previous examples are both special cases of this general result $F = \dfrac{\rho g h^2 (t+2b)}{6}$:

Example 8.5.1 (rectangle): $t = b = 5,\ h = 2 \quad \Rightarrow \quad F = \dfrac{\rho g \times 4(5+10)}{6} = 10\rho g$

Example 8.5.2 (apex-down triangle):
$t = 6,\ b = 0,\ h = 4 \quad \Rightarrow \quad F = \dfrac{\rho g \times 16(6+0)}{6} = 16\rho g$

END OF CHAPTER 8
