Formula Sheet for the Final Examination

1. Review of Calculus

$$(u \cdot v)' = u'v + uv'$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(a^{u(x)}\right) = u'(x)\, a^{u(x)} \ln a$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\frac{d}{dx}(\sin x) = \cos x\,, \quad \frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cos x) = -\sin x\,, \quad \frac{d}{dx}(\cosh x) = +\sinh x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x\,, \quad \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\sin(2\theta) = 2\sin\theta \cdot \cos\theta$$

$$\ln(u \cdot v) = \ln(u) + \ln(v)$$

$$\ln\left(x^n\right) = n \ln(x)$$

$$\int \frac{u'(x)}{u(x)}\, dx = \ln u(x) + C$$

$$\int \sin^2\theta\, d\theta = \frac{1}{2}\int \left(1 - \cos(2\theta)\right) d\theta$$

$$= \frac{2\theta - \sin(2\theta)}{4} + C$$

$$\int \sec x\, dx = \ln\left(\sec x + \tan x\right) + C$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\operatorname{Arctan}\left(\frac{x}{a}\right) + C$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}\left(a \sin bx - b\cos bx\right) + C$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}\left(a \cos bx + b\sin bx\right) + C$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\frac{d}{dx}\left(u(x)\right)^n = n\left(u(x)\right)^{n-1} u'(x)$$

$$\frac{d}{dx}\left(\ln\left(u(x)\right)\right) = \frac{u'(x)}{u(x)}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\ln\left(\frac{u}{v}\right) = \ln(u) - \ln(v)$$

$$\int u'(x) \cdot \left(u(x)\right)^n dx = \frac{\left(u(x)\right)^{n+1}}{n+1} + C$$

$$(n \neq -1)$$

$$\int \cos^2\theta\, d\theta = \frac{1}{2}\int \left(1 + \cos(2\theta)\right) d\theta$$

$$= \frac{2\theta + \sin(2\theta)}{4} + C$$

$$\int \csc x\, dx = \ln\left(\csc x - \cot x\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \operatorname{Arcsin}\left(\frac{x}{a}\right) + C$$

$$\int x^n \ln x\, dx = \frac{(n+1)\ln x - 1}{(n+1)^2}\, x^{n+1} + C \quad (n \neq -1)$$

2. Parametric and Polar Curves

$r^2 = x^2 + y^2$, $\tan\theta = \dfrac{y}{x}$, $x = r\cos\theta$, $y = r\sin\theta$

Full set of polar coordinates: $(r,\ \theta + 2n\pi)$ and $(-r,\ \theta + (2n+1)\pi)$ $(n \in \mathbb{Z})$

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3. Conic Sections

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$: Discriminant $\Delta = B^2 - 4AC$

If the conic section is not degenerate, then

$\Delta < 0$ $\Rightarrow$ ellipse (circle if also $B = 0$ and $C = A$)

$\Delta = 0$ $\Rightarrow$ parabola

$\Delta > 0$ $\Rightarrow$ hyperbola (rectangular if also $C = -A$)

If also $B = 0$ then

$A = C$ $\Rightarrow$ circle, otherwise

A, *C* same sign $\Rightarrow$ ellipse

one of *A*, *C* zero $\Rightarrow$ parabola

A, *C* opposite sign $\Rightarrow$ hyperbola

Standard form:

Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $b^2 = a^2(1 - e^2)$, $0 < e < 1$, centre (0, 0), foci $(\pm ae, 0)$,

directrices $x = \pm\dfrac{a}{e}$, vertices $(\pm a, 0)$

Parabola $y^2 = 4ax$, $e = 1$, focus $(a, 0)$, directrix $x = -a$, vertex $(0, 0)$

Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $b^2 = a^2(e^2 - 1)$, $e > 1$, centre (0, 0), foci $(\pm ae, 0)$,

directrices $x = \pm\dfrac{a}{e}$, vertices $(\pm a, 0)$, asymptotes $y = \pm\dfrac{bx}{a}$

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4. Quadric Surfaces

Ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, [prolate spheroid for $b = c < a$, oblate spheroid for $b = c > a$]

Hyperboloid of one sheet $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$, Hyperboloid of two sheets $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

Elliptic paraboloid $\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$, Hyperbolic paraboloid $\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

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5. Parametric Vector Functions

$$\vec{\mathbf{T}} = \left[\frac{dx}{dt}\ \frac{dy}{dt}\ \frac{dz}{dt}\right]^{\mathrm{T}} = \frac{d\vec{\mathbf{r}}}{dt}, \quad \frac{ds}{dt} = \left|\frac{d\vec{\mathbf{r}}}{dt}\right| \qquad \text{Arc length } L = \int_a^b \frac{ds}{dt}\,dt$$

Arc length along $y = f(x)$: $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Arc length along $r = f(\theta)$: $L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$, Area swept out $= \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$

Equation of surface of revolution of $y = f(x)$ about $y = c$: $(y-c)^2 + z^2 = \left(f(x)-c\right)^2$

Curved surface area $= 2\pi\int_a^b \left|f(x)-c\right|\sqrt{1+\left(f'(x)\right)^2}\,dx$

Volume enclosed $= \pi\int_a^b \left(f(x)-c\right)^2 dx$

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6. Series

"Race to Infinity": $n^n > n! > a^n\ (a>1) > n^c\ (c>0) > \log_b n\ (b>1)$
for all sufficiently large n.

Geometric series $S = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}\quad (|r|<1)$, diverges otherwise.

p series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges iff $p>1$.

Alternating p series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^p}$:

abs. conv. for $p>1$, cond. conv. for $0<p\le 1$, divergent for $p\le 0$

Divergence test: $\lim_{n\to\infty} a_n \ne 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

Comparison test:

$0 \le a_n \le b_n\ \forall n$ and $\sum b_n$ converges $\Rightarrow \sum a_n$ converges.

$a_n \ge b_n \ge 0\ \forall n$ and $\sum b_n$ diverges $\Rightarrow \sum a_n$ diverges.

Limit Comparison Test

$\lim_{n\to\infty}\frac{a_n}{b_n} = L \ge 0$ and $\sum b_n$ converges $\Rightarrow \sum a_n$ converges.

$\lim_{n\to\infty}\frac{a_n}{b_n} \ne 0$ and $\sum b_n$ diverges $\Rightarrow \sum a_n$ diverges.

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6. Series (continued)

Alternating Series Test

If $\dfrac{a_{n+1}}{a_n} < 0 \quad \forall n$ and $\lim\limits_{n\to\infty} a_n = 0$ then $\sum\limits_{n=1}^{\infty} a_n$ converges.

For alternating series, $| S - S_n | \leq | a_{n+1} |$

Ratio test:

If $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| < 1$ then $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent.

If $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| > 1$ then $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

Power series $f(x) = \sum\limits_{n=0}^{\infty} a_n (x-c)^{bn+d}$: radius of convergence $R = \lim\limits_{n\to\infty} \sqrt[b]{\left| \dfrac{a_n}{a_{n+1}} \right|}$

abs. conv. for $c - R < x < c + R$

Taylor series: $f(x) = \sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(c)}{n!}(x-c)^n$; remainder term $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!} | x-c |^{n+1}$

Maclaurin series: $f(x) = \sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!} x^n$;

$e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$; $\cos x = \sum\limits_{k=0}^{\infty} \dfrac{(-1)^k}{(2k)!} x^{2k}$; $\sin x = \sum\limits_{k=0}^{\infty} (-1)^k \dfrac{x^{2k+1}}{(2k+1)!}$; $\ln(1+x) = \sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k+1} x^k}{k}$

Binomial series: $(1+x)^n = 1 + \sum\limits_{k=1}^{\infty} \dfrac{n(n-1)(n-2)\ldots(n-(k-1))}{k!} x^k \qquad (-1 < x < 1)$

The Fourier series of $f(x)$ on the interval $(-L, L)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad (n = 0, 1, 2, 3, \ldots)$$

and

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad (n = 1, 2, 3, \ldots)$$

6. Series (continued)

Fourier sine series on (0, L):

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \text{ where } b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad (n = 1, 2, 3, \ldots)$$

Fourier cosine series on (0, L):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \text{ where } a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad (n = 0, 1, 2, 3, \ldots)$$

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7. Partial Differentiation

$$z = f(x,y) \;\Rightarrow\; dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \quad \text{and} \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

If $f(x,y,u,v) = c_1$ and $g(x,y,u,v) = c_2$ then the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)} = \left|\dfrac{\det B}{\det A}\right|$, where

$$A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -f_u & -f_v \\ -g_u & -g_v \end{pmatrix}$$

Spherical polar coordinates:

$$\vec{\mathbf{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{bmatrix} \;\Rightarrow\; dV = r^2 \sin\theta \, dr\, d\theta\, d\phi$$

Cylindrical polar coordinates:

$$\vec{\mathbf{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{bmatrix} \;\Rightarrow\; dV = \rho\, d\rho\, d\phi\, dz$$

Plane polar coordinates:

$$\vec{\mathbf{r}} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \;\Rightarrow\; dA = r\, dr\, d\theta$$

Directional derivative: $D_{\hat{\mathbf{a}}}F\big|_{P_o} = \vec{\nabla}F\big|_{P_o} \bullet \hat{\mathbf{a}}$

Normal line and tangent plane to surface $F(x,y,z) = c$ at $P(x_o, y_o, z_o)$ are

$$\frac{x - x_o}{A} = \frac{y - y_o}{B} = \frac{z - z_o}{C} \quad \text{and} \quad Ax + By + Cz = D, \text{ where } \begin{bmatrix} A \\ B \\ C \end{bmatrix} \text{ is parallel to } \vec{\nabla}F\big|_P$$

and $D = Ax_o + By_o + Cz_o$ (modify line equations if any of A, B, C are zero).

7. Partial Differentiation (continued)

Critical points where

1) P is on any boundary of domain D; or
2) $f(x_o, y_o)$ is undefined; or
3) f_x and/or f_y is undefined at P; or
4) f_x and f_y are both zero at P ($\Rightarrow \vec{\nabla} f = \vec{\mathbf{0}}$ at P).

Second derivative test: At each critical point where $\vec{\nabla} f = \vec{\mathbf{0}}$ evaluate $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$;

$D(a,b) > 0$ and $f_{xx}(a,b) > 0 \Rightarrow$ a relative minimum of f is at (a, b)

$D(a,b) > 0$ and $f_{xx}(a,b) < 0 \Rightarrow$ a relative maximum of f is at (a, b)

$D(a,b) < 0 \Rightarrow$ a saddle point of f is at (a, b)

$D(a,b) = 0 \Rightarrow$ test fails (no information).

Lagrange multipliers:

to find critical points of $f(x_1, x_2, \ldots, x_n)$ subject to a constraint $g(x_1, x_2, \ldots, x_n) = k$, solve

$$\begin{aligned} \vec{\nabla} f &= \lambda \vec{\nabla} g \\ g &= k \end{aligned}$$

For two constraints, solve

$$\begin{aligned} \vec{\nabla} f &= \lambda \vec{\nabla} g + \mu \vec{\nabla} h \\ g &= k \\ h &= c \end{aligned}$$

8. Multiple Integration

Second moments of area about the coordinate axes are

$$I_x = \iint_R y^2 \, dA \quad \text{and} \quad I_y = \iint_R x^2 \, dA$$

The second moments are minimized when both axes pass through the centroid.

Circle radius a: $I_x = I_y = \dfrac{\pi a^4}{4}$

Rectangle, base [parallel to x axis] b, height h: $I_x = \dfrac{bh^3}{12}$ and $I_y = \dfrac{b^3 h}{12}$

Triangle, base [parallel to x axis] b, height h: $I_x = \dfrac{bh^3}{36}$

Parallel axis theorem: $I_{x'} = I_x + d^2 A$

where the x axis passes through the centroid of the region of area A and d is the distance between the axes.