ENGI 4430 Final Examination

2019 Spring

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1. Find the rate of change of the function $z = f(x, y) = x^3 + y^2$ at the point $P(1, -1, 2)$ when one is moving in the direction of the vector $\vec{\mathbf{v}} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$ (that is, find the directional derivative $D_{\vec{\mathbf{v}}} f\big|_P = \vec{\nabla} f\big|_P \bullet \hat{\mathbf{v}}$). [12]

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2. A vector field $\vec{\mathbf{v}}$ in $\mathbb{R}^3$ is defined by

$$\vec{\mathbf{v}} = 2y\,\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

(a) Find the divergence $\vec{\nabla} \bullet \vec{\mathbf{v}}$ and the curl $\vec{\nabla} \times \vec{\mathbf{v}}$. [4]

(b) Find the equations of the family of lines of force (streamlines) for this vector field by any valid method. [7]

(c) Find the equations of the line of force that passes through the point $(1, -2, 5)$. [2]

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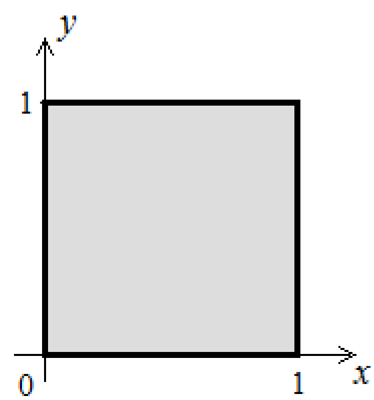
3. A thin metal plate in the shape of a square is as shown. It has a length of 1 m and a surface density

$$\sigma = 2x\left(\text{kg m}^{-2}\right)$$



(a) Verify that the centre of mass is at $(\bar{x}, \bar{y}) = \left(\frac{2}{3}, \frac{1}{2}\right)$ [9]

(b) Find the second moment of mass I_y about the vertical axis $x = \frac{2}{3}$ (through the centre of mass). [6]

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4. A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^3$ is defined in spherical polar coordinate system by

$$\vec{\mathbf{F}} = r\sin\theta\left(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}\right)$$

(a) Find the divergence of $\vec{\mathbf{F}}$ $\left(\vec{\nabla}\bullet\vec{\mathbf{F}}\right)$. [3]

(b) Show that $\operatorname{curl}\vec{\mathbf{F}} = \vec{\mathbf{0}}$ everywhere (except possibly the z axis). [3]

(c) For any simply-connected domain Ω in $\mathbb{R}^3$ that excludes the z-axis, find a potential function V such that $\vec{\nabla}V = \vec{\mathbf{F}}$ everywhere in Ω (or prove that no such potential function exists). [5]

(d) By any valid method find the work done by $\vec{\mathbf{F}}$ from the point $P(r,\theta,\phi) = \left(4,\frac{\pi}{6},0\right)$ to the point $Q(r,\theta,\phi) = \left(4,\frac{\pi}{2},\frac{\pi}{2}\right)$ along the path [4]

$$C:\ r = 4,\ \theta = \frac{\pi}{6}(1+2t),\ \phi = \frac{\pi t}{2} \quad (0 \le t \le 1).$$

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5. By any valid method, find the flux Φ through the simple closed surface S whose equation in parametric form is given by [15]

$$\vec{\mathbf{r}}(\theta,\phi) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6\sin\theta\cos\phi \\ 4\sin\theta\sin\phi \\ 5\cos\theta \end{bmatrix}, \quad (0 \le \theta \le \pi,\ 0 \le \phi < 2\pi)$$

due to the vector field $\vec{\mathbf{F}} = z\,\hat{\mathbf{k}}$.

BONUS QUESTION: [+5]

Prove that the flux Φ through *any* simple closed surface S due to the vector field $\vec{\mathbf{F}} = ax\,\hat{\mathbf{i}} + by\,\hat{\mathbf{j}} + (1-a-b)z\,\hat{\mathbf{k}}$ (for *any* constants a, b) is equal to the volume V enclosed by S.

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6. For the partial differential equation

$$3\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial^2 u}{\partial y^2} = 0$$

(a) Classify the partial differential equation as one of elliptic, parabolic or hyperbolic. [2]

(b) Find the general solution. [5]

(c) Find the complete solution, given the additional information [8]

$$u(x,0) = 0, \qquad u_y(x,0) = x$$

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7. The transverse displacement $y(x,t)$ of a point at time t and location x on an ideal wire whose ends are fixed at $x=0$ and $x=2$ is governed by the partial differential equation [15]

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad \text{with } c=3.$$

At time $t=0$ the displacement of the wire is $y(x,0)=2x-x^2$ and it is released from rest. Find the displacement $y(x,t)$ at all subsequent times. Write down the general term of the series for $y(x,t)$ ***and*** write down the first three non-zero terms.

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