ENGI 4430 Final Examination

2020 Spring

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1. A vector field $\vec{\mathbf{v}}$ in $\mathbb{R}^3$ is defined by

$$\vec{\mathbf{v}} = e^y\,\hat{\mathbf{i}} + 2x\,\hat{\mathbf{j}}$$

(a) Find the divergence $\vec{\nabla} \bullet \vec{\mathbf{v}}$ and the curl $\vec{\nabla} \times \vec{\mathbf{v}}$. [4]

(b) Find the equations of the family of lines of force (streamlines) for this vector field by any valid method. [7]

(c) Find the equations of the line of force that passes through the point $(-2, 0, 7)$. [2]

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2. A particle is following a path given in polar parametric form by [14]

$$r(t) = 1 - e^{-t}, \quad \theta(t) = \omega t \quad (t > 0)$$

where t is the time and ω is a constant.
Find the radial and transverse components of the velocity
and the radial and transverse components of the acceleration.
What path does the particle approach as $t \to \infty$?

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3. In the cylindrical polar coordinate system, a vector field $\vec{\mathbf{F}}$ is defined as

$$\vec{\mathbf{F}} = \cos\phi\,\hat{\boldsymbol{\rho}} - \sin\phi\,\hat{\boldsymbol{\phi}} + z\,\hat{\mathbf{k}}$$

on a simply connected domain that excludes the z axis.

(a) Find $\vec{\nabla} \times \vec{\mathbf{F}}$. [4]

(b) Find the potential function V for this vector field, (or show that no potential function exists). [5]

(c) Find the work done by $\vec{\mathbf{F}}$ from the point $P : (\rho, \phi, z) = (1, 0, 0)$ to the point $Q : (\rho, \phi, z) = \left(1, \tfrac{\pi}{2}, 4\right)$ along the helical path $C :\ \rho = 1,\ \phi = \tfrac{\pi}{2}t,\ z = 4t \quad (0 < t < 1)$. [5]

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4. A wire has a shape given by the Cartesian parametric equation

$$\vec{\mathbf{r}}(t) = \begin{bmatrix} \cos 2t \\ 2t + \sin 2t \\ 4\cos t \end{bmatrix} \quad \left(0 \le t \le \frac{\pi}{2}\right)$$

and a line density $\rho = \dfrac{t}{100}$ $\left(\text{kg m}^{-1}\right)$.

(a) Show that the derivative of the arc length s with respect to the parameter t is [5]

$$\frac{ds}{dt} = 4$$

(b) Find the length of the wire. [3]

(c) Find the z coordinate $\left(\bar{z}\right)$ of the centre of mass of the wire. [7]

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5. The transverse displacement $y(x,t)$ of a point at time t and location x on an ideal wire whose ends are fixed at $x = 0$ and $x = 4$ is governed by the partial differential equation [14]

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad \text{with} \quad c = 5 .$$

At time $t = 0$ the displacement of the wire is $y(x,0) = x^2 - 4x$ and it is released from rest. Find the displacement $y(x,t)$ at all subsequent times.

Write down the general term of the series for $y(x,t)$

and write down the first three non-zero terms.

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6. For the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x\,\partial y} + 2\frac{\partial^2 u}{\partial y^2} = -14y$$

(a) Classify the partial differential equation as one of elliptic, parabolic or hyperbolic. [2]

(b) Find the general solution. [5]

(c) Find the complete solution, given the additional information [8]

$$u(0, y) = 0 , \qquad u_x(0, y) = 3y^2$$

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7. For the vector field $\vec{\mathbf{F}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} + xy\,\hat{\mathbf{k}}$ find, by any valid method, the value of $I = \iint_S \vec{\nabla} \times \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{S}}$, where S is the hemisphere $x^2 + y^2 + z^2 = 9, \quad z \ge 0$ [15+5]

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