ENGI 4430 Final Examination
2021 Spring

1. Find the rate of change of the function $f(x,y,z) = x^2 - y^3 - z$ at the point $P(2,-1,2)$ when one is moving in the direction of the vector $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 12\hat{\mathbf{k}}$ (that is, find the directional derivative $D_{\vec{\mathbf{v}}} f\big|_P = \vec{\nabla} f\big|_P \bullet \hat{\mathbf{v}}$). [12]

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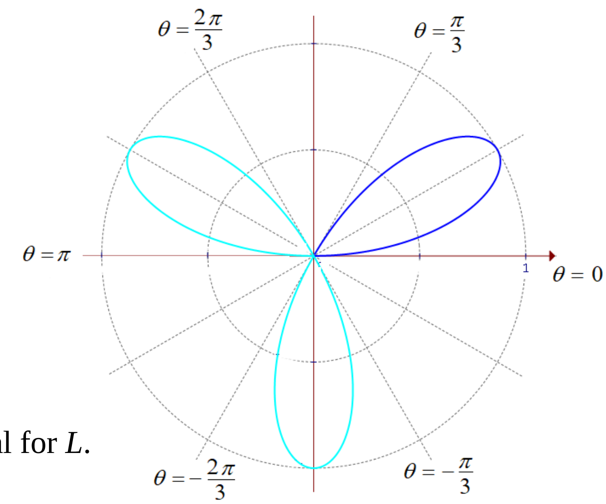
2. For the curve whose equation in polar coordinates is

$$r = \sin(3\theta)$$



(a) show that the arc length L along one loop of the curve (as illustrated) can be expressed as [6]

$$L = \int_0^{\pi/3} \sqrt{1 + 8\cos^2(3\theta)}\, d\theta$$

Do ***not*** attempt to evaluate this integral for L.

(b) find the exact value of the area A enclosed by one loop. [6]

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3. A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^3$ (except on the z axis) is defined in spherical polar coordinates by

$$\vec{\mathbf{F}} = \frac{1}{r^2}\left(\sin\theta\,\hat{\mathbf{r}} - \cos\theta\,\hat{\boldsymbol{\theta}}\right)$$

(a) Find $\operatorname{div}\vec{\mathbf{F}}$ everywhere (except the z-axis). [4]

(b) Find $\operatorname{curl}\vec{\mathbf{F}}$ everywhere (except the z-axis). [4]

(c) For any simply-connected domain Ω in $\mathbb{R}^3$ that excludes the z-axis, find a potential function V, such that $\vec{\mathbf{F}} = \vec{\nabla} V$ everywhere in Ω. [6]

(d) Find the work done by $\vec{\mathbf{F}}$ in moving from the point P $(r,\theta,\phi) = \left(2, \frac{\pi}{2}, 0\right)$ to the point Q $(r,\theta,\phi) = \left(2, \frac{\pi}{6}, 0\right)$. [4]

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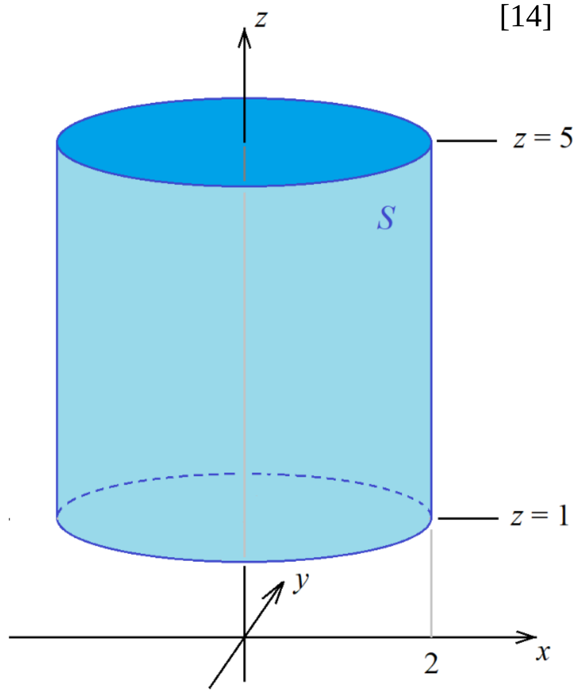
4. By any valid method, find the flux Φ passing out from the simple closed surface *S*, which is a right circular cylinder, centred on the *z* axis, of radius 2 and height 4 (between *z* = 1 and *z* = 5), (as shown here), due to the vector field $\vec{\mathbf{F}}$, given in cylindrical polar coordinates by [14]

$$\vec{\mathbf{F}} = \rho\,\hat{\boldsymbol{\rho}} + (z-5)\,\hat{\mathbf{k}}$$



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5. The transverse displacement $y(x,t)$ of a point at time t and location x on an ideal wire whose ends are fixed at $x = 0$ and $x = 1$ is governed by the partial differential equation [12]

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad \text{with } c = 3.$$

At time $t = 0$ the displacement of the wire is $y(x,0) = \sin(\pi x)$ and it is released from rest. Find the displacement $y(x, t)$ at all subsequent times.

[You may quote $\displaystyle\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} \dfrac{L}{2} & (m=n) \\ 0 & (m \neq n) \end{cases}$

for all positive integers *m*, *n*.]

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6. For the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x\,\partial y} + 5\frac{\partial^2 u}{\partial y^2} = 8y - 4x$$

(a) Classify the partial differential equation as one of elliptic, parabolic or hyperbolic. [2]

(b) Find the general solution. [7]

(c) Find the complete solution, given the additional information [10]

$$u(0,y) = 0, \qquad u_x(0,y) = -y^2$$

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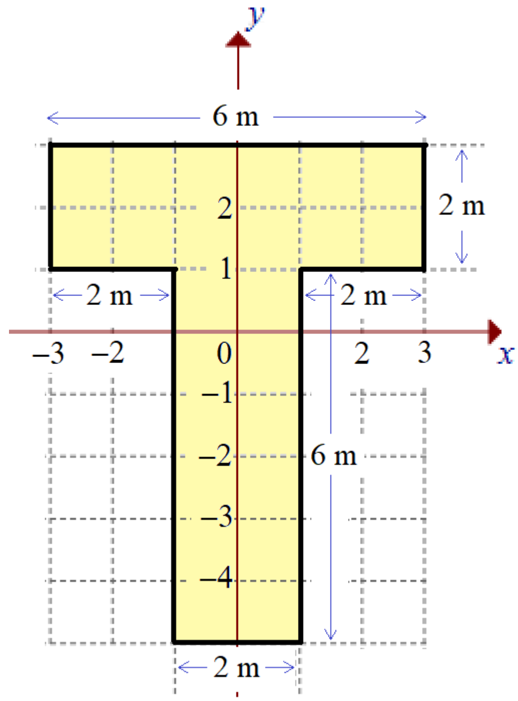
7. A thin sheet of metal of constant surface density σ is in the shape shown.



(a) Show that the centre of mass is at the origin $\left((\bar{x}, \bar{y}) = (0,0) \right)$ [5]

(b) Find the second moments of mass I_x and I_y and hence the moment of inertia *I*, in terms of the mass *m*. [8+5]

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