ENGI 4430 Advanced Calculus for Engineering
Faculty of Engineering and Applied Science

Problem Set 6 Questions

[Non-Cartesian coordinates]

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1. Convert from Cartesian coordinates to cylindrical polar coordinates the vector

$$\vec{\mathbf{F}} = (x-y)\,\hat{\mathbf{i}} + (x+y)\,\hat{\mathbf{j}} + e^{-\left(x^2+y^2\right)}\hat{\mathbf{k}}$$

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2. The coordinate conversion matrix A for Cartesian coordinates to spherical polar coordinates is

$$A = \begin{bmatrix} s_\theta c_\phi & s_\theta s_\phi & c_\theta \\ c_\theta c_\phi & c_\theta s_\phi & -s_\theta \\ -s_\phi & c_\phi & 0 \end{bmatrix}, \qquad \text{where} \quad \begin{matrix} c_\theta = \cos\theta & s_\theta = \sin\theta \\ c_\phi = \cos\phi & s_\phi = \sin\phi \end{matrix}$$

so that $\vec{\mathbf{F}}_{\text{sph}} = A\vec{\mathbf{F}}_{\text{cart}}$, where

$$\vec{\mathbf{F}}_{\text{sph}} = \begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix} = F_r\,\hat{\mathbf{r}} + F_\theta\,\hat{\boldsymbol{\theta}} + F_\phi\,\hat{\boldsymbol{\phi}} \quad \text{and} \quad \vec{\mathbf{F}}_{\text{cart}} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\,\hat{\mathbf{k}}$$

Show that the conversion matrix B for the inverse conversion from spherical polar back to Cartesian coordinates, (such that $\vec{\mathbf{F}}_{\text{cart}} = B\vec{\mathbf{F}}_{\text{sph}}$), is simply $B = A^{\mathrm{T}}$ (the transpose of matrix A).

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3. Convert from spherical polar to Cartesian coordinates the vector field

$$\vec{\mathbf{F}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

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4. For the vector field defined in spherical polar coordinates by

$$\vec{\mathbf{u}}(r,\theta,\phi) = \left(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}\right)\sin\phi \;+\; \cos\phi\,\hat{\boldsymbol{\phi}}$$

find $\dfrac{d\vec{\mathbf{u}}}{dt}$.

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5. Consider the purely radial vector field $\vec{\mathbf{F}}(r,\theta,\phi) = f(r)\,\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the unit radial vector in the spherical polar coordinate system and $f(r)$ is any function of r that is differentiable everywhere in $\mathbb{R}^3$ (except possibly at the origin).

(a) Find an expression, in terms of r, $f(r)$ and $f'(r)$, for the divergence of $\vec{\mathbf{F}}$.
(b) Find an expression, in terms of r, $f(r)$ and $f'(r)$, for the curl of $\vec{\mathbf{F}}$.
(c) Of particular interest is the central force law

$$\vec{\mathbf{F}} = \frac{k}{r^n}\,\hat{\mathbf{r}}, \quad (k > 0,\; r > 0)$$

Show that the divergence of $\vec{\mathbf{F}}$ vanishes everywhere in $\mathbb{R}^3$ (except possibly at the origin) if and only if $n = 2$.
[Two of the four fundamental forces of nature, electromagnetism and gravity, both obey this inverse square law.]

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6. The displacement vector is $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} = r\,\hat{\mathbf{r}}$

(a) Find the divergence of $\vec{\mathbf{r}}$ using the Cartesian form $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$.
(b) Verify the value of div $\vec{\mathbf{r}}$ using the spherical polar form $\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$.

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7. In the evaluation of an area integral $\iint_A f(\vec{\mathbf{r}})\,dA$ or a volume integral $\iiint_V f(\vec{\mathbf{r}})\,dV$, the Cartesian differentials are related to the differentials of a different parameterization (u, v, w) of the surface or volume by the Jacobian determinant:

$$dA = dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \quad \text{or} \quad dV = dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw,$$

where the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix} \quad \text{or} \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{bmatrix}.$$

Evaluate the Jacobian in order to determine the conversion formula from Cartesian differentials $dx\,dy$ (or $dx\,dy\,dz$) to

(a) Plane polar area differentials $dr\,d\theta$
(b) Cylindrical polar volume differentials $d\rho \;\; d\phi \;\; dz$
(c) Spherical polar volume differentials $dr\,d\theta\,d\phi$
(d) Parabolic area differentials $du\,dv$, where $x = uv$, $y = v^2 - u^2$.
(e) In part (d), sketch on the same x-y plane the coordinate curves $u = \frac{1}{2}$, $u = 1$, $u = 2$, $v = \frac{1}{2}$, $v = 1$ and $v = 2$.
(f) Hence find the area in the first quadrant bounded by the parabolas $y = 4x^2 - \frac{1}{4}$, $y = x^2 - 1$, $y = \frac{1}{4} - 4x^2$ and $y = 1 - x^2$.

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8. The displacement vector of a particle, in spherical polar coordinates, is $\vec{\mathbf{r}}(t) = r\,\hat{\mathbf{r}}$, with $r(t) = 2$, $\theta(t) = \frac{\pi}{3}$, $\phi(t) = 2t$.

(a) Find the velocity $\vec{\mathbf{v}}(t)$.
(b) What path does the particle follow?

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