ENGI 4430 Advanced Calculus for Engineering
Faculty of Engineering and Applied Science

Problem Set 9 Questions

[Theorems of Gauss and Stokes]

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1. A flat area A is bounded by the triangle C whose vertices are the points $P(0, 0, 1)$, $Q(1, 0, 1)$ and $R(1, 1, 1)$.
[Note that this triangle is entirely in the plane $z = 1$.]

(a) Show that the unit normal $\hat{\mathbf{n}}$ to A that points away from the origin is $\hat{\mathbf{n}} = \hat{\mathbf{k}}$.

(b) A vector field $\vec{\mathbf{F}}$ is defined in $\mathbb{R}^3$ by $\vec{\mathbf{F}} = y^2\,\hat{\mathbf{i}} + x^2\,\hat{\mathbf{j}} + (z - x)\hat{\mathbf{k}}$.
Show that $\text{curl } \vec{\mathbf{F}} = \hat{\mathbf{j}} + 2(x - y)\hat{\mathbf{k}}$.

(c) Find the circulation $I = \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ of $\vec{\mathbf{F}}$ around C.
[Hint: you may use Stokes' theorem.]

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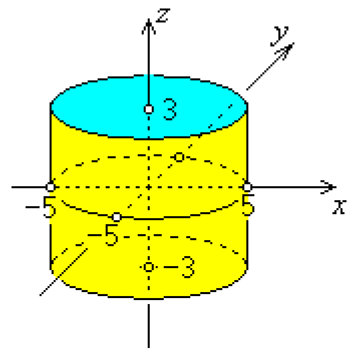
2. A vector field $\vec{\mathbf{F}}$ is defined in the cylindrical polar coordinate system in $\mathbb{R}^3$ by

$$\vec{\mathbf{F}} = \frac{\rho}{5}\hat{\boldsymbol{\rho}}$$

Find the total flux $\Phi = \oiint_S \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}}$ due to $\vec{\mathbf{F}}$ out through the right circular cylinder S of radius 5, height 6, aligned along the z-axis with the top and bottom ends at $z = +3$ and $z = -3$, as shown.



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3. An extended source of electric charge has a charge density

$$\rho(r) = \begin{cases} r\,e^{-r} & (r \le 2) \\ 0 & (r > 2) \end{cases}$$

where r = the distance of the point (x, y, z) from the origin.

(a) Find the total charge Q due to this extended object.

(b) Find the total flux due to the extended charge through the simple closed surface S defined by $(x-3)^2 + (y-4)^2 + z^2 = 1$.

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4. Calculate the circulation of $\vec{\mathbf{F}} = \begin{bmatrix} x-y & x^2 y & x^3 y^2 z\, e^{xyz} \end{bmatrix}^T$ counterclockwise around the unit circle in the *xy*-plane. [*Hint*: Use Stokes' theorem, letting the surface S be any smooth surface that has C as its boundary.]

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5. Complete the evaluation (in Example 10.04 of the lecture notes) of the line integral $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ around the unit square in the *x*-*z* plane for

$\vec{\mathbf{F}} = \begin{bmatrix} xyz & xz & e^{xy} \end{bmatrix}^T$, (without using Stokes' theorem).

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6. A vector field $\vec{\mathbf{F}}$ is defined in the spherical polar coordinate system in $\mathbb{R}^3$ by

$$\vec{\mathbf{F}} = r\, e^{-r}\, \hat{\mathbf{r}}$$

(a) Find the divergence $\vec{\nabla} \bullet \vec{\mathbf{F}}$ in spherical polar coordinates.

(b) Find the total flux $\Phi = \oiint_S \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}}$ due to $\vec{\mathbf{F}}$ out through the sphere S of radius 2, centre at the origin.

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