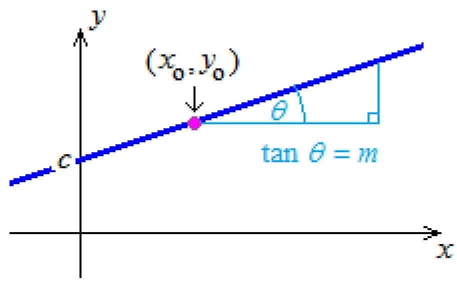
Standard curves

Straight lines

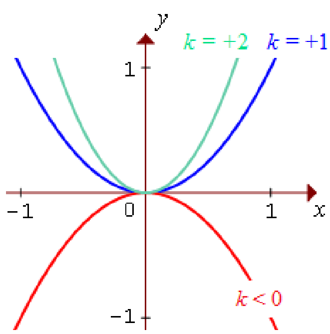
$y = mx + c$

$(y - y_o) = m(x - x_o)$

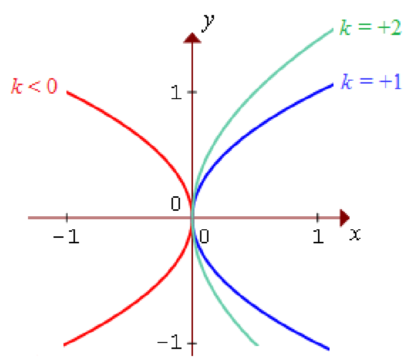


Polynomials

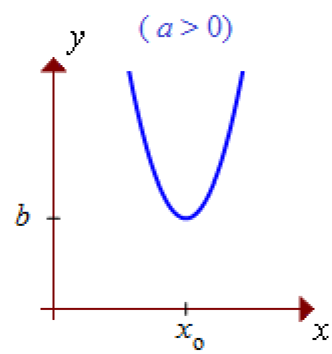
$y = kx^2$



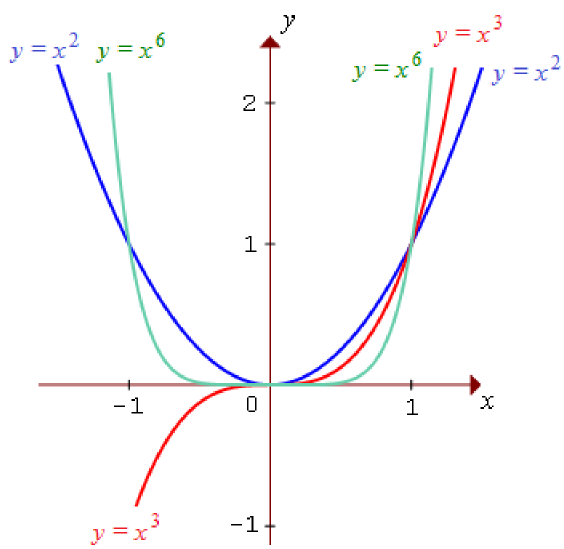
$y^2 = kx$



$y = a(x - x_o)^2 + b$



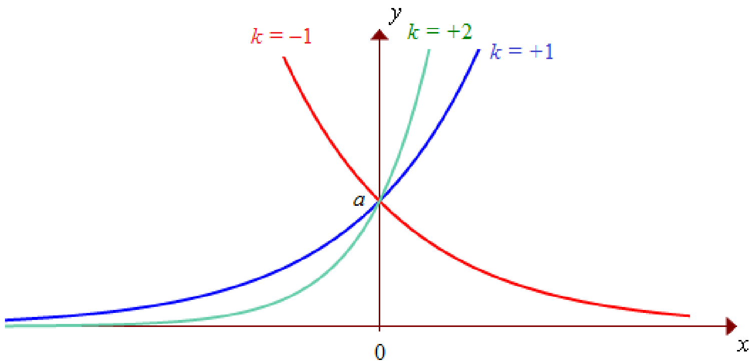
$y = kx^3, \quad y = kx^4, \qquad$ etc.

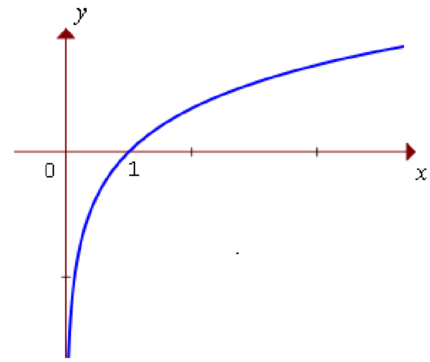


The function $y = kx^n$ is even (the graph is symmetric in the *y*-axis) when the exponent *n* is even.

The function $y = kx^n$ is odd (the graph is in diametrically opposite quadrants) when the exponent *n* is odd.

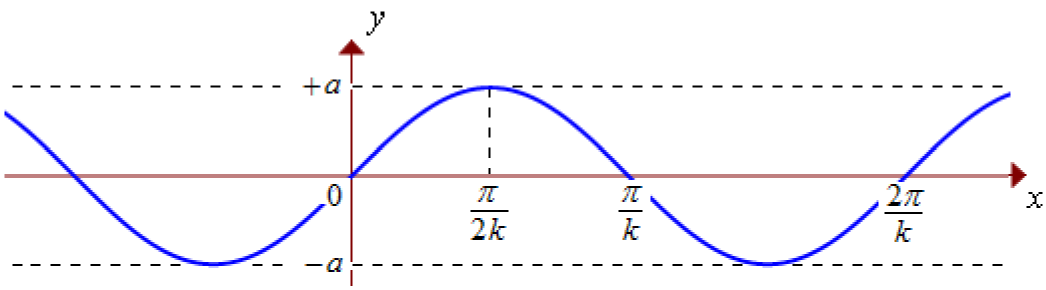
Exponentials $y = ae^{kx}$ (exponential decay if $k < 0$) $y = \ln x$



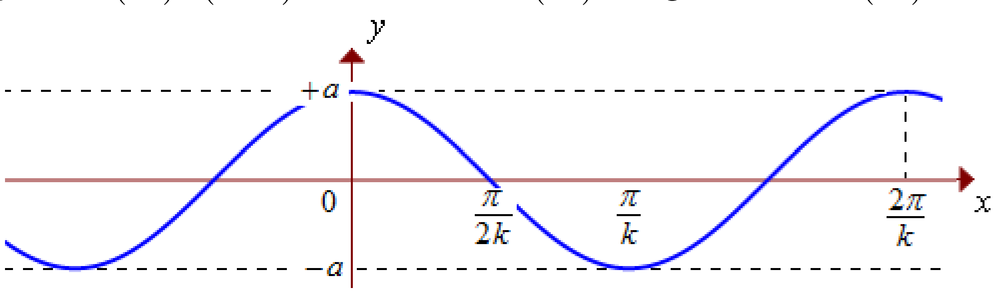


Trigonometric functions

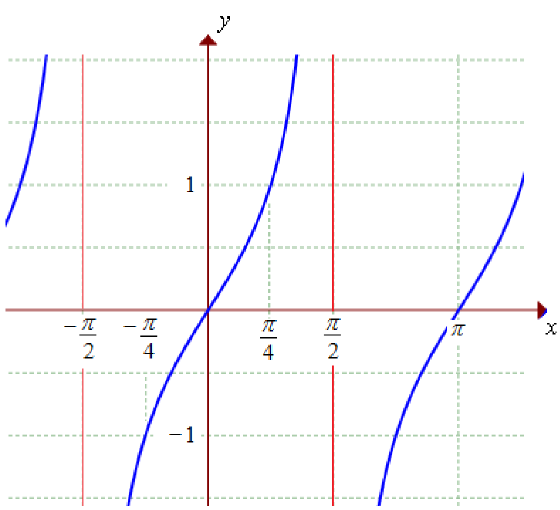
$y = a\sin(kx)$ $(a > 0,\ k > 0)$ [$\sin(kx)$ first returns to zero at $(kx) = \pi$]



$y = a\cos(kx)$ $(a > 0)$ [$\cos(kx)$ first goes to zero at $(kx) = \pi/2$]



$y = \tan x$



$|\tan x| \to \infty$ at odd multiples of $\frac{\pi}{2}$

<u>**Conic Sections**</u>

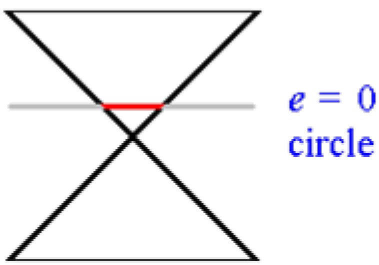
All members of the family of curves known as conic sections can be generated, (as the name implies), from the intersections of a plane and a double cone. The Cartesian equation of any conic section is a second order polynomial in x and y. If each axis of symmetry is parallel to a coordinate axis, then the Cartesian equation is of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

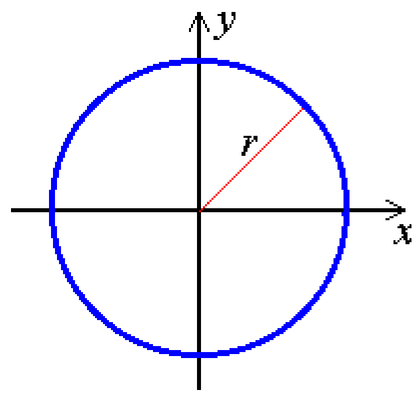
where A, C, D, E and F are constants. There is no "xy" term, so $B = 0$.
If the curve passes through the origin, then $F = 0$.
If the centre of the curve is at the origin, then $D = E = 0$ [impossible for the parabola].

The slope of the intersecting plane is related to the **eccentricity** e of the conic section.

Circle



$$x^2 + y^2 = r^2$$

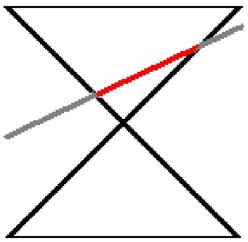


or, if the centre is at (h, k), $(x-h)^2 + (y-k)^2 = r^2$

A parametric form for a circle, centre at the origin, radius r is

$$(x, y) = (r\cos\theta,\ r\sin\theta), \qquad (0 \le \theta < 2\pi).$$

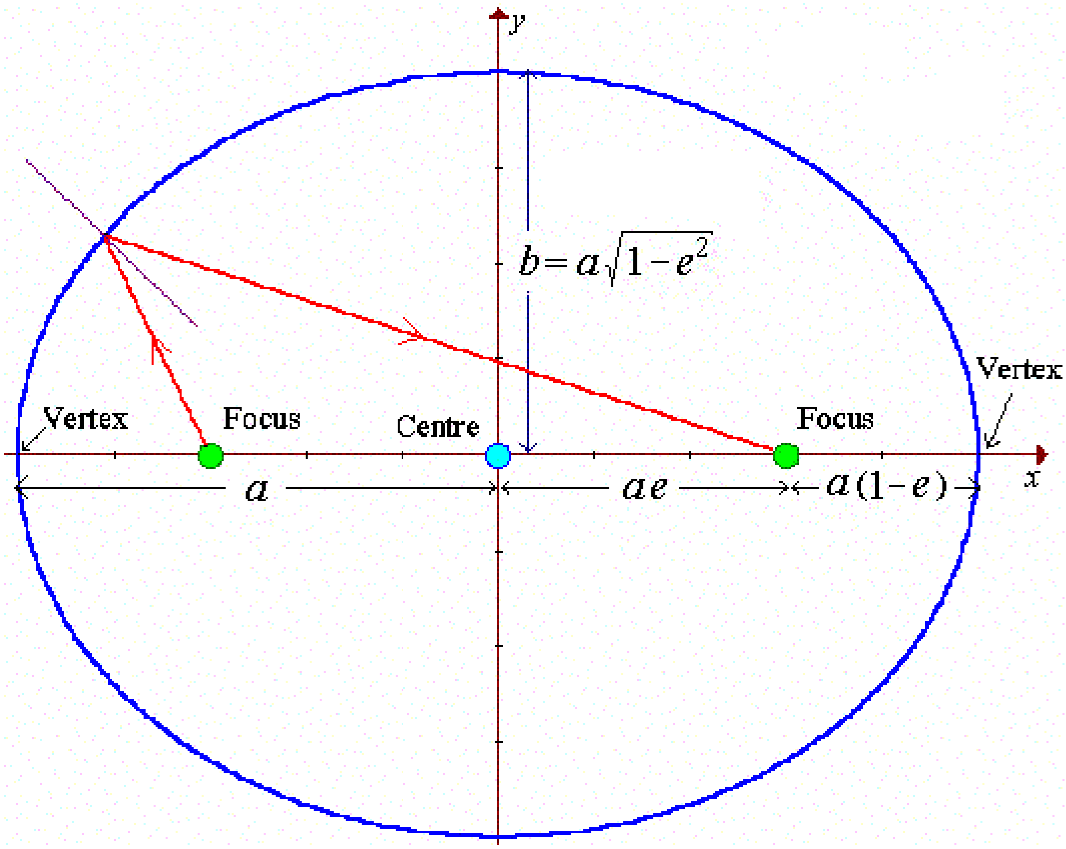
Ellipse



Cartesian equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $b^2 = a^2\left(1 - e^2\right)$



The circle is clearly a special case of the ellipse, with $e = 0$ and $b = a = r$.

The longest diameter is the major axis ($2a$). The shortest diameter is the minor axis ($2b$).

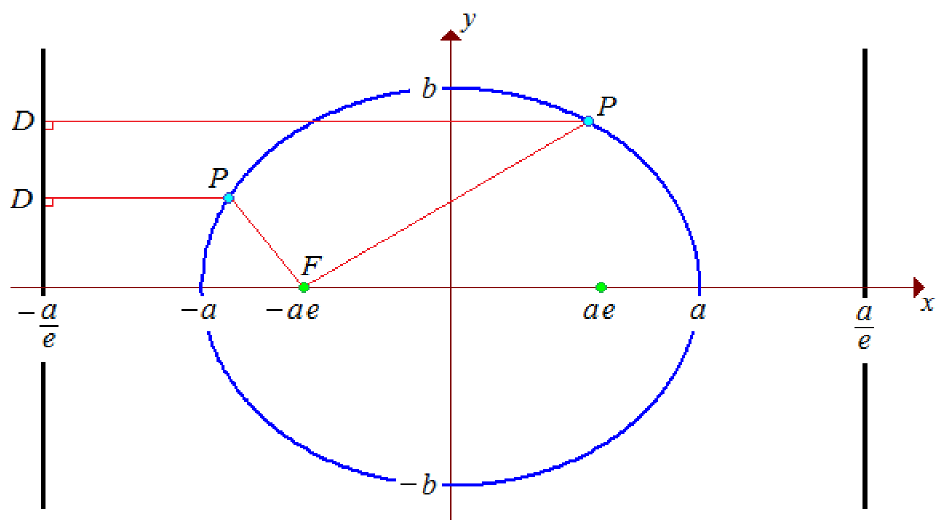
If a mirror is made in the shape of an ellipse, then all rays emerging from one focus will, after reflection, converge on the other focus.

A parameterization for the ellipse is $\vec{\mathbf{r}}(\theta) = a\cos\theta\,\hat{\mathbf{i}} + b\sin\theta\,\hat{\mathbf{j}}, \quad (0 \le \theta < 2\pi)$.

Ellipse (continued)

The ellipse also possesses directrices at $x = \pm\frac{a}{e}$, parallel to the minor axis.

Let P be any point on the ellipse, let F be a focus and let D be point on the directrix nearer to F such that the line segment PD is parallel to the major axis.



Then the eccentricity e is defined as $\quad e = \frac{PF}{PD}$

The ellipse shown here has an eccentricity $e = 0.6$.

The directrices of a circle are at infinity.

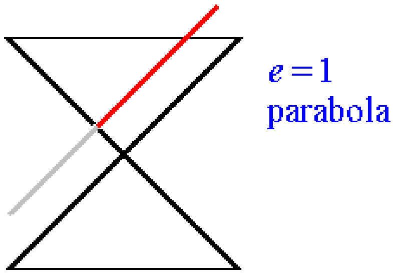
Another feature of the ellipse is that the sum of the distances to the foci is constant for all points P on the ellipse.

$$PF_1 + PF_2 = 2a$$

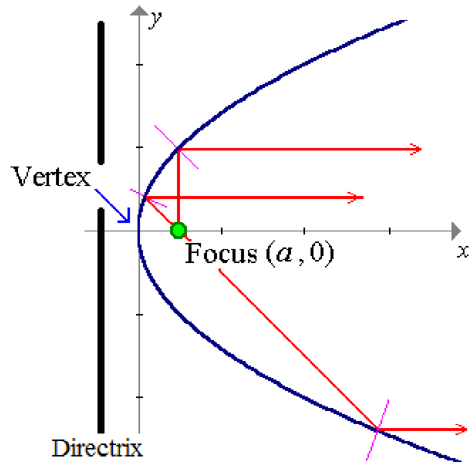
If the ellipse is translated so that the centre moves from $(0,0)$ to (h,k) then the equation changes to

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Parabola



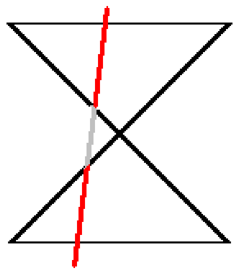
$y^2 = 4ax$

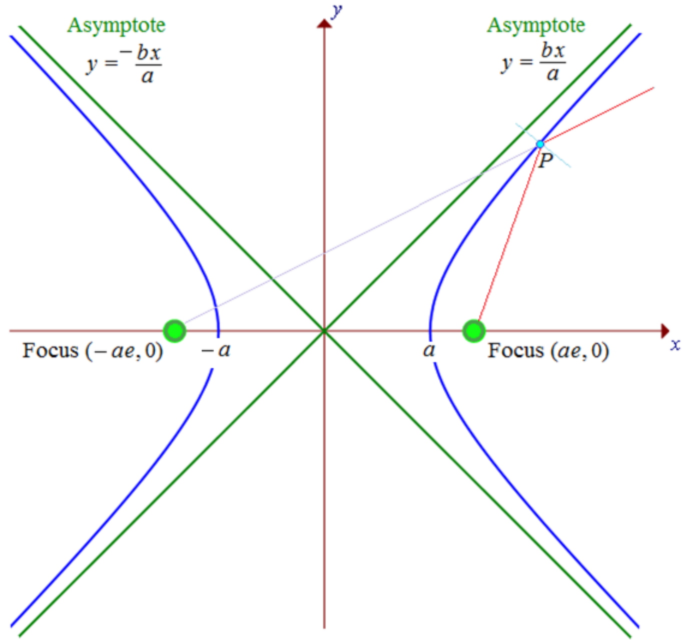


One vertex is at the origin, one directrix is at $x = -a$
and one focus is at $(a, 0)$.
The centre and the other vertex, focus and directrix are at infinity.

If a mirror is made in the shape of a parabola, then all rays emerging from the focus will, after reflection, travel in parallel straight lines to infinity (where the other focus is). The primary mirrors of most telescopes follow a paraboloid shape.

Hyperbola





$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b^2 = a^2\left(e^2 - 1\right)$

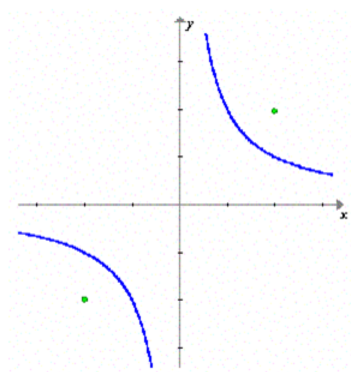
The hyperbola has two separate branches. As the curve retreats towards infinity, the curve approaches the asymptotes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \quad \left(\Rightarrow \quad y = \pm\frac{bx}{a}\right).$$

The distance between the two vertices is the major axis ($2a$).
If a mirror is made in the shape of an hyperbola, then all rays emerging from one focus will, after reflection, appear to be diverging from the other focus.

Circles and ellipses are closed curves. Parabolas and hyperbolas are open curves.

A special case of the hyperbola occurs when the eccentricity is $e = \sqrt{2}$ and it is rotated $45°$ from the standard orientation. The asymptotes line up with the coordinate axes, the graph lies entirely in the first and third quadrants and the Cartesian equation is $xy = k$.



This is the **rectangular hyperbola**.

Degenerate conic sections arise when the intersecting plane passes through the apex of the cone. Two cases are:

$0 \le e < 1$: $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0 \quad$ point at the origin.

$e > 1$: $\quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0 \quad$ line pair through the origin.

Another degenerate case is

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1 \quad$ nothing! [no real (x, y) can satisfy this equation]

Curve shifts

For the curve $y = f(x)$

$y = f(x+a)$ translates the curve to the left by a

$y = f(ax)$ compresses the curve horizontally around the y-axis by a factor of a

$y = f(x) + b$ translates the curve up by b

$y = b \cdot f(x)$ stretches the curve vertically around the x-axis by a factor of b

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<u>Example</u>:

The graph of $y = 2\sin\left(3x + \dfrac{\pi}{6}\right) + 1$ is the same as the graph of $y = \sin x$ after it has been

- compressed horizontally by a factor of 3, then
- translated to the left by $\dfrac{\pi}{6}$, then
- stretched vertically by a factor of 2, then
- translated up by 1.