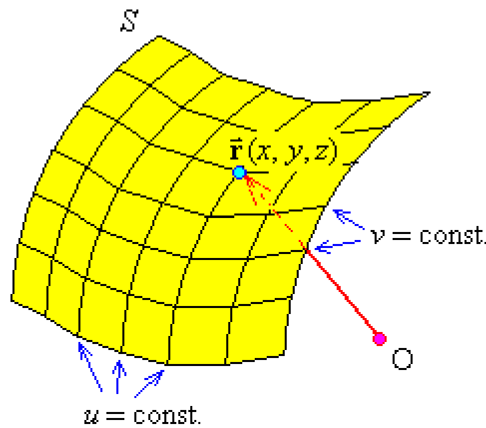
1. Parametric Vector Functions

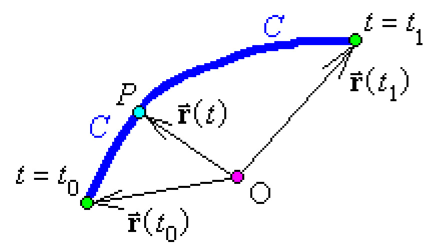
The general position vector $\vec{\mathbf{r}}(x, y, z)$ has its tail at the origin (0, 0, 0) and its tip at any point (x, y, z) in $\mathbb{R}^3$.

One constraint (that is, one equation) connecting x, y and z usually causes the loss of one degree of freedom.

The resulting position vector $\vec{\mathbf{r}}\big(x(u,v), y(u,v), z(u,v)\big)$ is free to move only on a two-dimensional surface, where u and v are the independent parameters of that surface. Specifying values of both parameters will identify a single point on that surface. Specifying values of one parameter will identify a curve on that surface. This establishes a coordinate grid on the surface S.



Two independent constraints connecting x, y and z usually cause the loss of two degrees of freedom. The resulting position vector $\vec{\mathbf{r}}\big(x(t), y(t), z(t)\big)$ is free to move only on a one-dimensional curve. Specifying the value of the parameter will identify a single point on that curve.



Similarly, one constraint in $\mathbb{R}^2$ results in the loss of one degree of freedom, so that the resulting position vector $\vec{\mathbf{r}}\big(x(t), y(t)\big)$ is free to move only on a one-dimensional curve. Specifying the value of the parameter will identify a single point on that curve.

A parameter may represent a physical quantity (such as time or angle), or it may be abstract, with no apparent physical interpretation. In MATH 2050 the concept of a parameter was introduced in the description of a family of solutions for a linear system of equations that has infinitely many solutions.

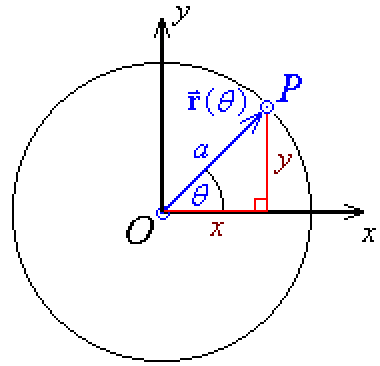
The **distance** of a particle from the origin at any value of t is given by the scalar function

$$r(t) = \left|\vec{\mathbf{r}}(t)\right| = \sqrt{\big(x(t)\big)^2 + \big(y(t)\big)^2 + \big(z(t)\big)^2}$$

Note the various alternative conventions for a vector and its magnitude:

$\mathbf{r} \equiv \vec{\mathbf{r}} \equiv \underline{\mathbf{r}}$ and $r \equiv \|\vec{\mathbf{r}}\| \equiv |\vec{\mathbf{r}}|$

Example 1.01



An obvious parameterization for a path $\vec{\mathbf{r}}(\theta) = \begin{bmatrix} x(\theta) & y(\theta) \end{bmatrix}^T$ on the circumference of a circle of radius a centre the origin is

$$\begin{bmatrix} x(\theta) \\ y(\theta) \end{bmatrix} =$$

The range of a parameter is assumed to be all real values, unless specified otherwise.

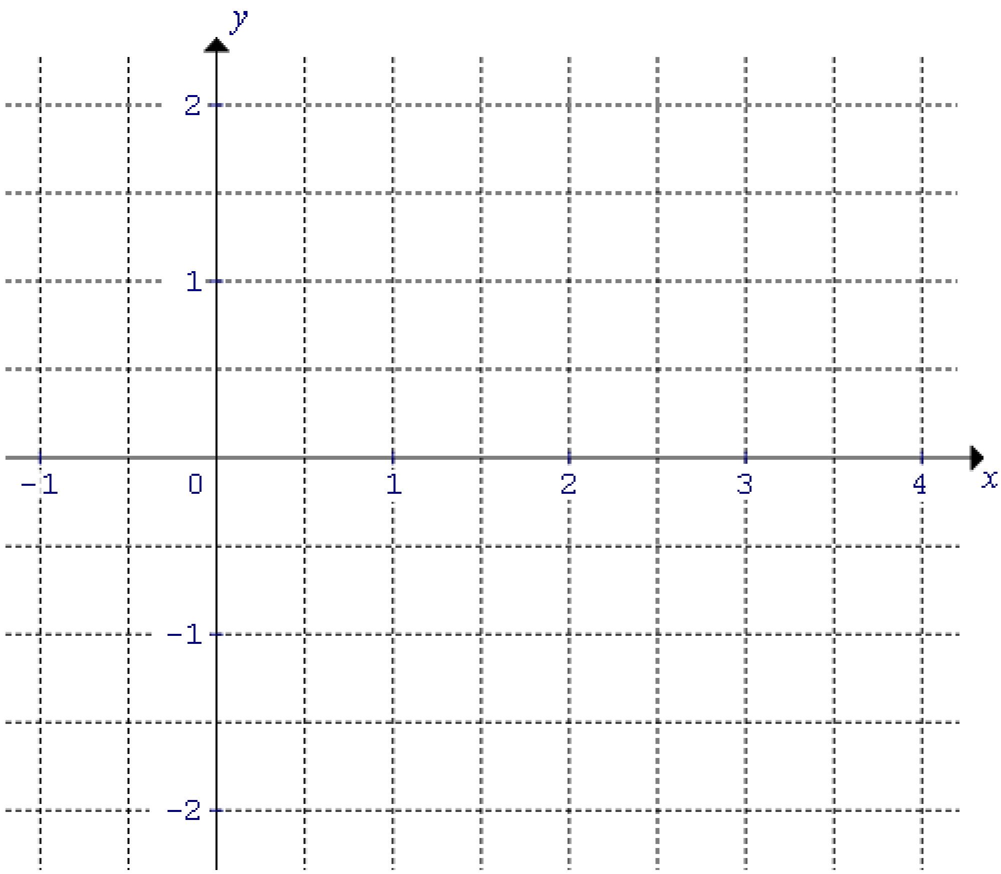
Example 1.02

Sketch the curve in $\mathbb{R}^2$ that is given in parametric form by

$$\vec{\mathbf{r}}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t^2 \\ t^3 - 3t \end{bmatrix}$$

Identify all values of the parameter t for which any of x, y or their derivatives with respect to t are zero.

Example 1.02 (continued)



In $\mathbb{R}^2$, horizontal tangents to a curve $\vec{\mathbf{r}}(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}^{\mathrm{T}}$ occur at points where

and vertical tangents occur at points where

At points where $\frac{dx}{dt} = \frac{dy}{dt} = 0$, the tangent line can have any slope, or no well defined slope at all. At such points the slope has to be determined by taking the appropriate limit.

Note that, by the chain rule,

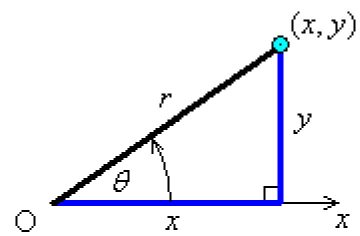
$$\frac{dy}{dx} =$$

Therefore, in Example 1.02 above, the slopes of the tangent lines on the curve in its two passes through the point (3, 0) are

$$m = \left.\frac{dy}{dx}\right|_{(3,0)} =$$

Polar Coordinates

The description of the location of an object in $\mathbb{R}^2$ relative to the observer is not very natural in Cartesian coordinates: "the object is three metres to the east of me and four metres to the north of me", or $(x, y) = (3, 4)$. It is much more natural to state how far away the object is and in what direction: "the object is five metres away from me, in a direction approximately 53° north of due east", or $(r, \theta) \approx (5, 53°)$.



This is the polar coordinate system (r, θ).

Note that the angle θ is undefined at the pole $(r = 0)$.

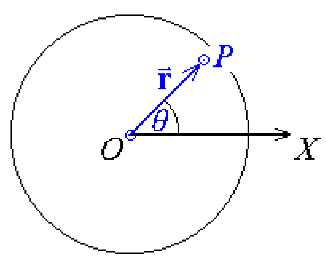
Radar also operates more naturally in plane polar coordinates.

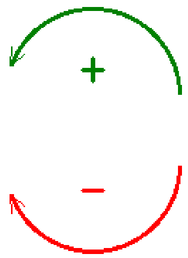
r = range

θ = azimuth

O is the pole

OX is the polar axis (where $\theta = 0$)



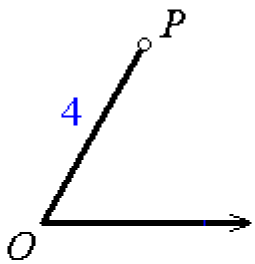


Anticlockwise rotations are positive.

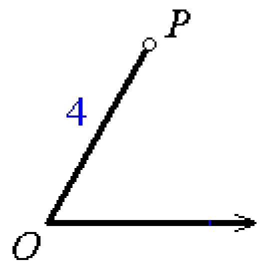
Every point in $\mathbb{R}^2$ has a unique representation (x, y) in Cartesian coordinates.
This is not true for polar coordinates.

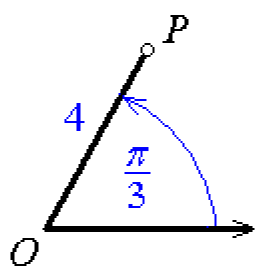
<u>Example 1.03</u>

The point P with the polar coordinates $(r, \theta) = \left(4, \frac{\pi}{3}\right)$
also has the polar coordinates



or





In general, if the polar coordinates of a point are (r, θ), then

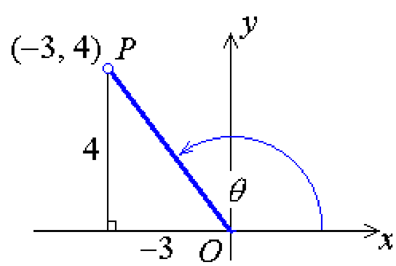
(n = any integer) also describe the same point.

The polar coordinates of the pole are $(0, \theta)$ for *any* θ.

In some situations, we impose restrictions on the range of the polar coordinates, such as $r \geq 0$, $-\pi < 0 \leq +\pi$ for the principal value of a complex number in polar form.

Example 1.04

Find the complete set of polar coordinates for the point whose Cartesian coordinates are (−3, 4).



The principal value of r is easy to find: $r = \sqrt{x^2 + y^2}$.

Note that if $x \le 0$, then the principal value of θ , $(-\pi < \theta \le \pi)$ is ***not*** $\arctan\left(\frac{y}{x}\right)$, because $-\frac{\pi}{2} \le \arctan\left(\frac{y}{x}\right) \le \frac{\pi}{2}$ (fourth and first quadrants only, for which $x > 0$).

When (x, y) is in the second quadrant, $\theta = \arctan\left(\frac{y}{x}\right) + \pi$ (as in Example 1.04).

When (x, y) is in the third quadrant, $\theta = \arctan\left(\frac{y}{x}\right) - \pi$.

Example 1.05

Find the Cartesian coordinates for $(r, \theta) = \left(2, -\frac{11\pi}{3}\right)$.

Polar Curves $\boldsymbol{r = f(\theta)}$

The representation (x, y) of a point in Cartesian coordinates is unique. For a curve defined implicitly or explicitly by an equation in x and y, a point (x, y) is on the curve if and only if its coordinates (x, y) satisfy the equation of the curve.

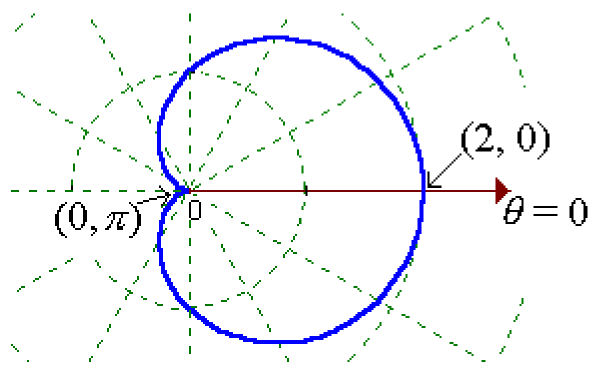
The same is *not* true for plane polar coordinates. Each point has infinitely many possible representations, $(r,\ \theta + 2n\pi)$ and $(-r,\ \theta + (2n+1)\pi)$ (where n is any integer). A point lies on a curve if and only if at least one pair (r, θ) of the infinitely many possible pairs of polar coordinates for that point satisfies the polar equation of the curve. It doesn't matter if other polar coordinates for that same point do not satisfy the equation of the curve.

<u>Example 1.06</u>

The curve whose polar equation is

$$r = 1 + \cos\theta$$

is a **cardioid**
(literally, a "heart-shaped" curve).



$\{\, r = 2,\ \ \theta = 2n\pi \,\}$
(where n is any integer)
satisfies the equation $r = 1 + \cos\theta$.

$\Rightarrow \quad (r,\theta) = (2,\, 2n\pi)$ is on the cardioid curve.

But $(2,\, 2n\pi)$ is the same point as $(-2,\ (2n+1)\pi)$.

$\theta = (2n+1)\pi \quad \Rightarrow$

Yet the point whose polar coordinates are $(-2,\ (2n+1)\pi)$ is on the curve!

Example 1.07

Convert to Cartesian form the equation of the cardioid curve $r = 1 + \cos\theta$.

Tangents to $r = f(\theta)$

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

By the chain rule for differentiation:

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

This leads to a general expression for the slope anywhere on a curve $r = f(\theta)$:

$$\boxed{\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

$\dfrac{dy}{d\theta} = 0$ and $\dfrac{dx}{d\theta} \neq 0$ at (r,θ) $\Rightarrow$

$\dfrac{dx}{d\theta} = 0$ and $\dfrac{dy}{d\theta} \neq 0$ at (r,θ) $\Rightarrow$

At the pole ($r = 0$):

$$\frac{dy}{dx} =$$

Example 1.08

Sketch the curve whose equation in Cartesian form is

$$x^2 + y^2 = \sqrt{x^2 + y^2} + 2y \ .$$

First convert the equation to polar form.

Note that there is no restriction on the sign of r ; it can be negative.

Example 1.08 (continued)

θ is the parameter.

Method 1 for sketching the polar curve:

$x = r\cos\theta =$

and

$y = r\sin\theta =$

Therefore the *y*-axis intercepts are at $(x, y) =$

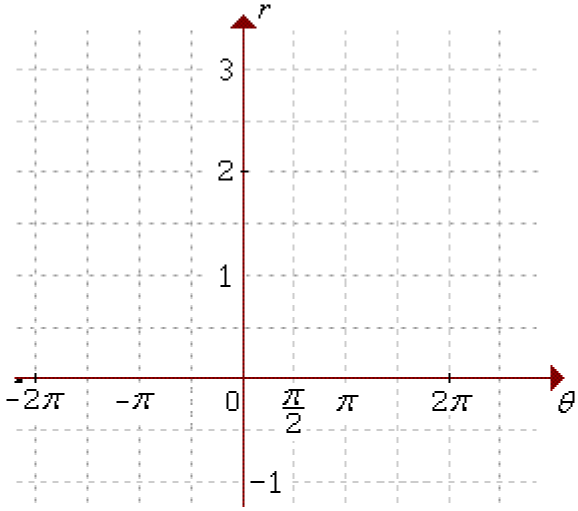
Therefore the *x*-axis intercepts are at $(x, y) =$

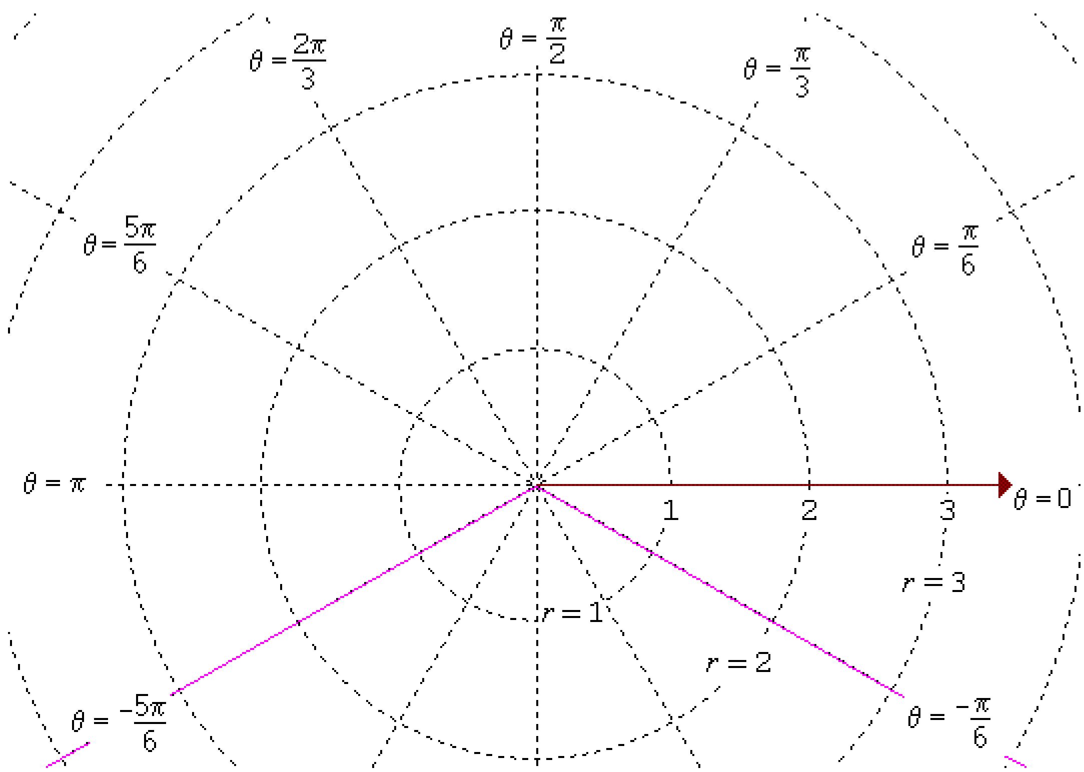
$r = 1 + 2\sin\theta \quad \Rightarrow$

Example 1.08 (continued)

Method 2

Sketch the graph of $r = 1 + 2\sin\theta$ as though r were Cartesian y and θ were Cartesian x, then transfer that Cartesian sketch onto a polar sketch.

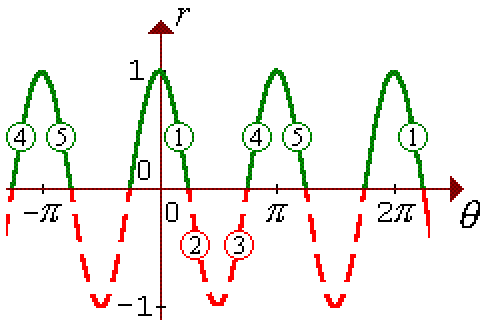


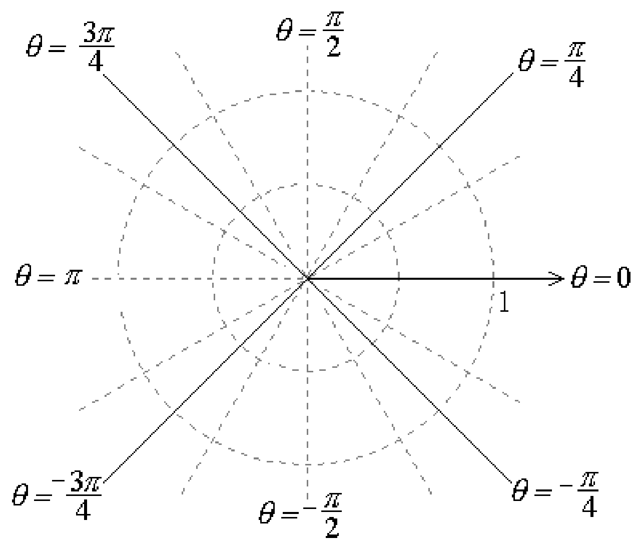


Example 1.09

Sketch the curve whose equation in polar form is $r = \cos 2\theta$.

Using Method 2,





You can follow a plot of $r = \cos n\theta$ by Method 2 (for n = 1, 2, 3, 4, 5 and 6) on the web site. See the link at "`http://www.engr.mun.ca/~ggeorge/4430/demos/`".

The distinct polar tangents of $r = \cos 2\theta$ are

[End of Chapter 1]