

## 2. Parametric Vector Functions (continued)

Any non-zero vector  $\vec{r}$  can be decomposed into its magnitude  $r$  and its direction:

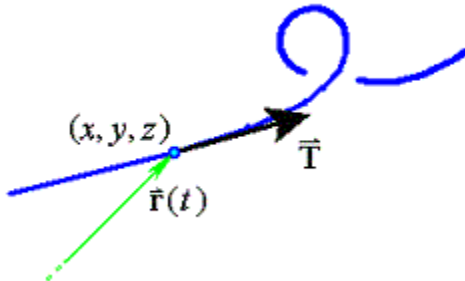
$$\vec{r} \equiv r\hat{\mathbf{r}}, \quad \text{where } r \equiv |\vec{r}| > 0$$

### Tangent Vector:

$$\vec{\mathbf{T}} = \left[ \frac{dx}{dt} \quad \frac{dy}{dt} \quad \frac{dz}{dt} \right]^T = \frac{d\vec{r}}{dt}$$

If the parameter  $t$  is time, then the tangent vector is also the **velocity vector**,

$$\vec{v} = \frac{d\vec{r}}{dt}, \quad \text{whose magnitude is the **speed** } v = \left| \frac{d\vec{r}}{dt} \right|.$$



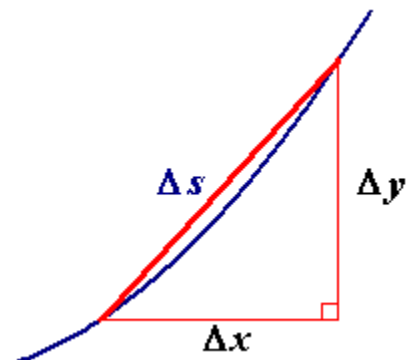
The unit tangent is

$$\mathbf{T} = \frac{d\vec{r}}{dt} \div \left| \frac{d\vec{r}}{dt} \right|$$

### Arc Length

In  $\mathbb{R}^2$ :

In  $\mathbb{R}^3$ :



The vector  $\frac{d\vec{r}}{dt}$  points in the direction of the tangent  $\vec{\mathbf{T}}$  to the curve defined parametrically by  $\vec{r} = \vec{r}(t)$ .

Example 2.01

(a) Find the arc length along the curve defined by

$\mathbf{r}(t) = \left[ \frac{1}{4}(2t - \sin 2t) \quad \frac{1}{4}\cos 2t \quad \sin t \right]^T$ , from the point where  $t = 0$  to the point where  $t = 4\pi$ .

(b) Find the unit tangent  $\hat{\mathbf{T}}$ .

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(a)

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### Arc Length for a Polar Curve

For a polar curve defined by  $r = f(\theta)$ ,  $(\alpha \leq \theta \leq \beta)$ , the parameter is  $\theta$  and

$$x = f(\theta)\cos\theta \quad y = f(\theta)\sin\theta.$$

Using the abbreviations  $r = f(\theta)$ ,  $r' = f'(\theta)$ ,  $c = \cos\theta$ ,  $s = \sin\theta$ ,

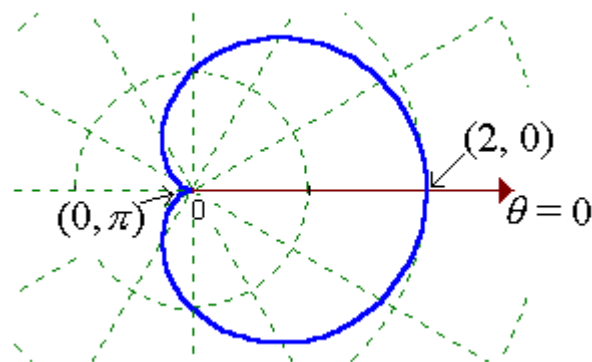
$$\frac{dx}{d\theta} = \quad \text{and} \quad \frac{dy}{d\theta} =$$

$$\Rightarrow \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 =$$

Therefore the arc length  $L$  along the polar curve  $r = f(\theta)$  from  $\theta = \alpha$  to  $\theta = \beta$  is

#### Example 2.02

Find the length  $L$  of the perimeter of the cardioid  $r = 1 + \cos\theta$



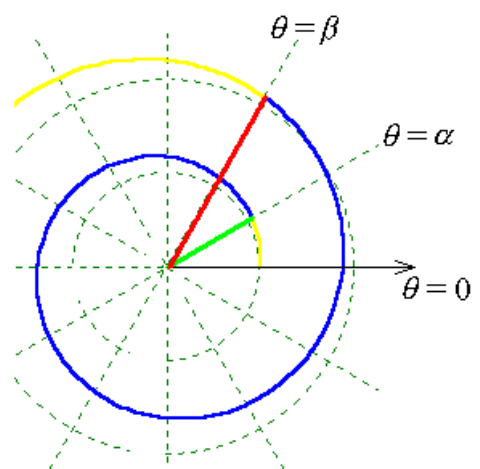
Example 2.02 (continued)

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Example 2.03

Find the arc length along the spiral curve  $r = ae^\theta$  ( $a > 0$ ), from  $\theta = \alpha$  to  $\theta = \beta$ .

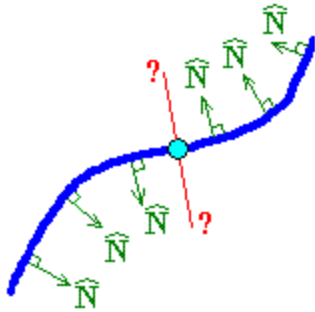
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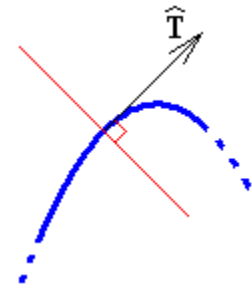
### Overview of Normal and Binormal:

For a curve in  $\mathbb{R}^2$ , there is a unique normal line at every point where the unit tangent vector is defined:

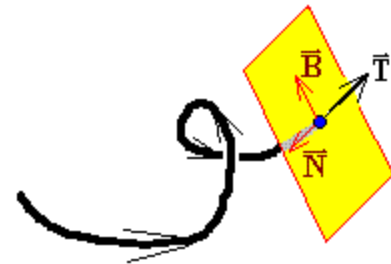
There are only two choices for the unit normal vector.



$\hat{N}$  is chosen to point in the direction in which the curve is turning. This definition fails wherever the unit tangent does not exist. This definition also fails at all points of inflexion and wherever the curve is a straight line.



However, for a curve in  $\mathbb{R}^3$ , there is a normal *plane* at each point, not a normal line. From the infinite number of different directions for a normal vector, we need to identify a unique direction for a principal normal. Again, we shall choose the direction in which the unit tangent vector is turning at each point.



The **unit binormal** is at right angles to both tangent and principal normal:

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$$

$$\hat{\mathbf{B}} = \text{constant} \Rightarrow$$

The three vectors form an orthonormal set (they are mutually perpendicular and each one is a unit vector; that is, magnitude = 1). All three unit vectors are undefined where the curve suddenly reverses direction, (such as at a cusp).

### Curvature:

The derivative of the unit tangent with respect to distance travelled along a curve is the principal normal vector,  $\bar{\mathbf{N}} = \frac{d\hat{\mathbf{T}}}{ds}$ . Its direction is the unit principal normal  $\hat{\mathbf{N}}$ . Its magnitude is a measure of how rapidly the curve is turning and is defined to be the **curvature**:

$$\kappa = |\bar{\mathbf{N}}| = \left| \frac{d\hat{\mathbf{T}}}{ds} \right| = \left| \frac{d\hat{\mathbf{T}}}{dt} \right| \div \left| \frac{d\bar{\mathbf{r}}}{dt} \right|$$

The **radius of curvature** is  $\rho = \frac{1}{\kappa}$ .

The radius of curvature at a point on a curve is the radius of the circle which best fits the curve at that point.

### Example 2.01 (continued)

(c) Find the curvature  $\kappa(s)$  at any point for which  $s > 0$ , for the curve

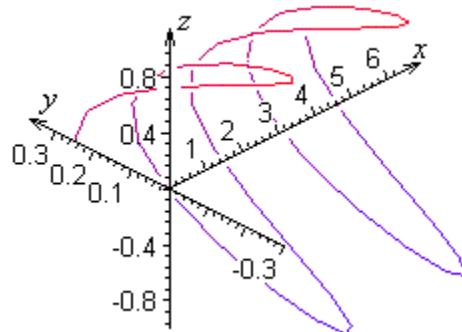
$$\bar{\mathbf{r}}(s) = \left[ \frac{1}{4}(2s - \sin 2s) \quad \frac{1}{4} \cos 2s \quad \sin s \right]^T$$

where  $s$  is the arc length from the point  $(0, \frac{1}{4}, 0)$ .

From part (b):

$$\hat{\mathbf{T}} = \begin{bmatrix} \sin^2 s \\ -\sin s \cos s \\ \cos s \end{bmatrix}$$

Example 2.01 (c) (continued)



A Maple plot of this curve is available from the demonstration files section of the web site,  
[www.engr.mun.ca/~ggeorge/4430/demos/index.html](http://www.engr.mun.ca/~ggeorge/4430/demos/index.html).

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**Another formula for curvature:**

Let  $\dot{\mathbf{r}} = \frac{d\bar{\mathbf{r}}}{dt}$  and  $\dot{s} = \frac{ds}{dt} = \left| \frac{d\bar{\mathbf{r}}}{dt} \right| = |\dot{\mathbf{r}}|$ , then

$$\frac{d\bar{\mathbf{r}}}{dt} = \left| \frac{d\bar{\mathbf{r}}}{dt} \right| \hat{\mathbf{T}} \Rightarrow \dot{\mathbf{r}} = \dot{s} \hat{\mathbf{T}}$$

$$\Rightarrow \ddot{\mathbf{r}} = \ddot{s} \hat{\mathbf{T}} + \dot{s} \frac{d\hat{\mathbf{T}}}{dt}$$

But  $\frac{d\hat{\mathbf{T}}}{dt} =$

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Example 2.04

Find the curvature and the radius of curvature for the helix, given in parametric form by  $x = \cos t$ ,  $y = \sin t$ ,  $z = t$ . Assume SI units.

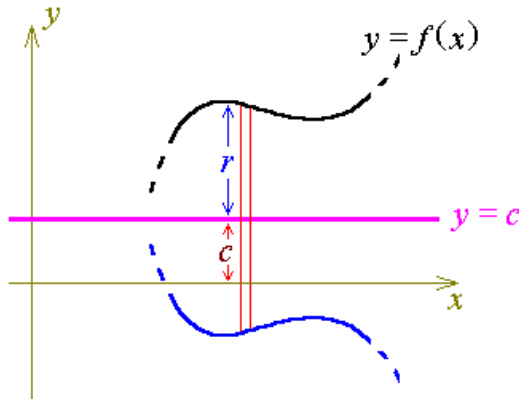
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Let  $c = \cos t$ ,  $s = \sin t$ .

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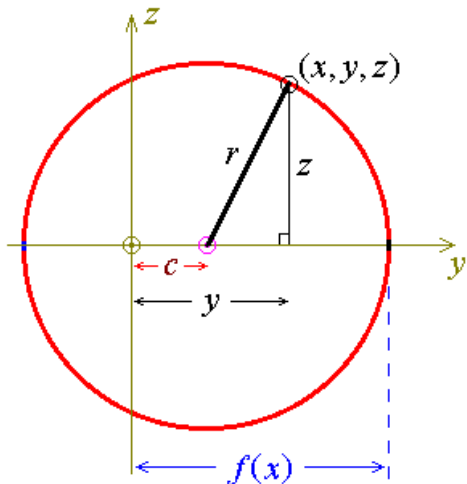
**Surfaces of Revolution**

Consider a curve in the  $x$ - $y$  plane, defined by the equation  $y = f(x)$ .  
 If it is swept once around the line  $y = c$ , then it will generate a surface of revolution.



At any particular value of  $x$ , a thin cross-section through that surface, parallel to the  $y$ - $z$  plane, will be a circular disc of radius  $r$ , where

Let us now view the circular disc face-on, (so that the  $x$  axis and the axis of rotation are both pointing directly out of the page and the page is parallel to the  $y$ - $z$  plane).



Let  $(x, y, z)$  be a general point on the surface of revolution.

From this diagram, one can see that

$$r^2 =$$

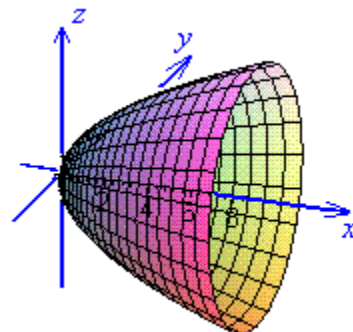
Therefore, the equation of the surface generated, when the curve  $y = f(x)$  is rotated once around the axis  $y = c$ , is

Special case: When the curve  $y = f(x)$  is rotated once around the  $x$  axis, the equation of the surface of revolution is

Example 2.05

Find the equation of the surface generated, when the parabola  $y^2 = 4ax$  is rotated once around the  $x$  axis.

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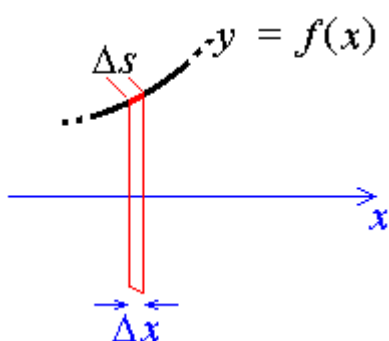


A Maple worksheet for this surface is available from the demonstration files section of the ENGI 4430 web site.

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**The Curved Surface Area of a Surface of Revolution**

For a rotation around the  $x$  axis,



the curved surface area swept out by the element of arc length  $\Delta s$  is approximately the product of the circumference of a circle of radius  $y$  with the length  $\Delta s$ .

Integrating along a section of the curve  $y = f(x)$  from  $x = a$  to  $x = b$ , the total curved surface area is

For a rotation of  $y = f(x)$  about the axis  $y = c$ , the curved surface area is

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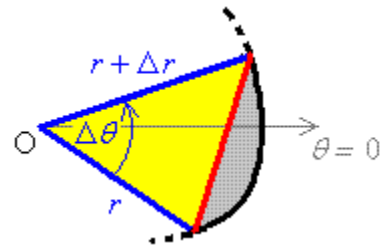
Example 2.06

Find the curved surface area of the circular paraboloid generated by rotating the portion of the parabola  $y^2 = 4cx$  ( $c > 0$ ) from  $x = a$  ( $\geq 0$ ) to  $x = b$  ( $> a$ ) about the  $x$  axis.

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$$A = 2\pi \int_{x=a}^{x=b} |y| ds$$

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**Area Swept Out by a Polar Curve  $r = f(\theta)$**  $\Delta A \approx$  Area of triangle

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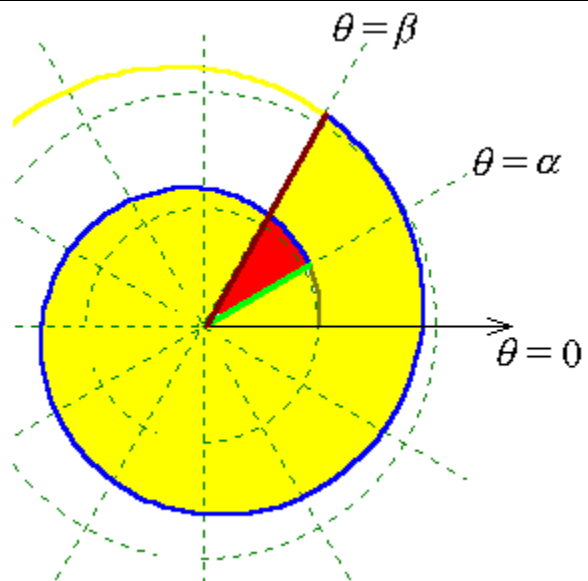
**Example 2.07**Find the area of a circular sector, radius  $r$ , angle  $\theta$ .

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Example 2.08

Find the area swept out by the polar curve  $r = ae^{\theta}$  over  $\alpha < \theta < \beta$ ,  
 (where  $a > 0$  and  $\alpha < \beta < \alpha + 2\pi$ ).

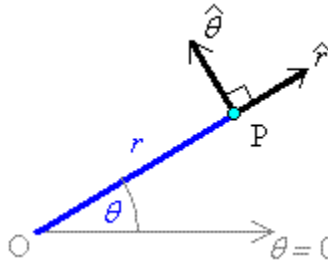
The condition  $(\alpha < \beta < \alpha + 2\pi)$   
 prevents the same area being swept  
 out more than once.



In general, the area bounded by two polar curves  $r = f(\theta)$  and  $r = g(\theta)$  and the radius vectors  $\theta = \alpha$  and  $\theta = \beta$  is

$$A = \frac{1}{2} \int_{\alpha}^{\beta} \left( (f(\theta))^2 - (g(\theta))^2 \right) d\theta$$

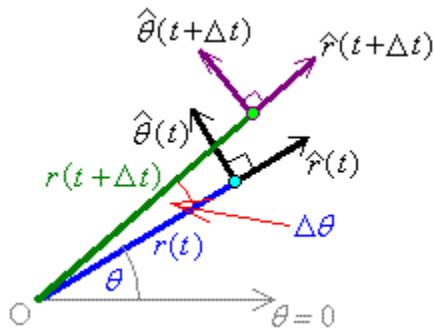
### Radial and Transverse Components of Velocity and Acceleration



At any point P (not at the pole), the unit radial vector  $\hat{\mathbf{r}}$  points directly away from the pole. The unit transverse vector  $\hat{\boldsymbol{\theta}}$  is orthogonal to  $\hat{\mathbf{r}}$  and points in the direction of increasing  $\theta$ . These vectors form an orthonormal basis for  $\mathbb{R}^2$ .

Unlike the Cartesian  $\hat{\mathbf{i}}$  and  $\hat{\mathbf{j}}$ , if  $\theta$  is not constant then  $\hat{\mathbf{r}}$  and  $\hat{\boldsymbol{\theta}}$  will be variable unit vectors.

Examine  $\hat{\mathbf{r}}$  and  $\hat{\boldsymbol{\theta}}$  at two nearby times  $t$  and  $t + \Delta t$ :

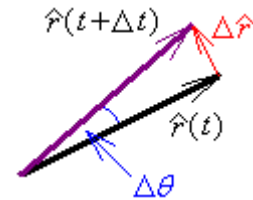


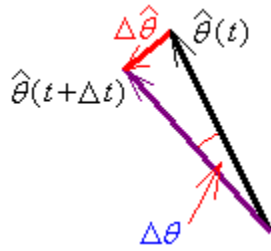
The magnitudes of the unit vectors can never change.

By definition, the magnitude of any unit vector is always 1.

Only the directions can change.

Examine the difference in the unit radial vector at these two times.





Therefore the derivatives of the two [non-constant] polar unit vectors are

$$\frac{d\hat{\mathbf{r}}}{dt} = \left(\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} \quad \text{and} \quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\left(\frac{d\theta}{dt}\right)\hat{\mathbf{r}}$$

Using the “overdot” notation to represent differentiation with respect to the parameter  $t$ , these results may be expressed more compactly as

$$\dot{\hat{\mathbf{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}} \quad \text{and} \quad \dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\mathbf{r}}$$

The radial and transverse components of velocity and acceleration then follow:

$$\bar{\mathbf{r}} = r\hat{\mathbf{r}} \Rightarrow \bar{\mathbf{v}} = \dot{\bar{\mathbf{r}}} =$$

$$\bar{\mathbf{a}} = \dot{\bar{\mathbf{v}}} = \ddot{\bar{\mathbf{r}}} =$$

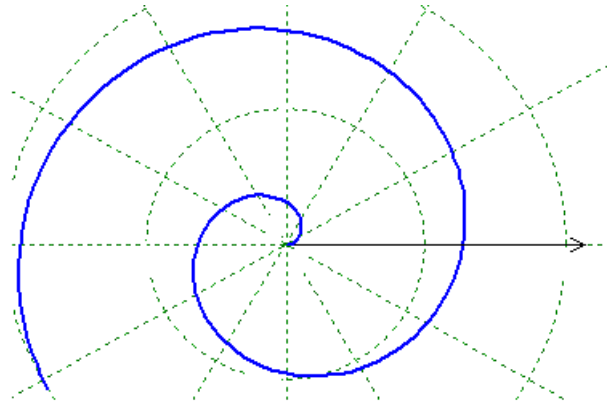
The transverse component of acceleration can also be written as



Example 2.09

A particle follows the path  $r = \theta$ , where the angle at any time is equal to the time:  
 $\theta = t > 0$ . Find the radial and transverse components of acceleration.

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Example 2.10

For circular motion around the pole, with constant radius  $r$  and constant angular velocity  $\dot{\theta} = \omega$ , the velocity vector is purely tangential,  $\bar{v} = r\omega\hat{\theta}$ , and the acceleration vector is

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### Tangential and Normal Components of Velocity and Acceleration

From page 2.01, we know that the velocity vector is also the tangent vector,  $\bar{\mathbf{v}} = \bar{\mathbf{T}} = \frac{d\bar{\mathbf{r}}}{dt}$ .

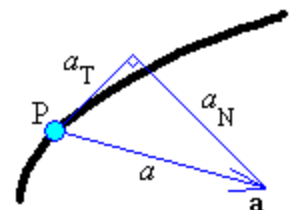
The [scalar] speed is  $v = |\bar{\mathbf{v}}| = \left| \frac{d\bar{\mathbf{r}}}{dt} \right| = \frac{ds}{dt}$

The speed is therefore the derivative with respect to time of distance travelled along the curve.

It follows that  $\bar{\mathbf{v}} = v\hat{\mathbf{T}}$  - the speed is also the tangential component of the velocity vector. There are no components of velocity in the normal directions.

The acceleration vector is

$$\bar{\mathbf{a}} = \frac{d\bar{\mathbf{v}}}{dt} = \frac{d}{dt}(v\hat{\mathbf{T}}) =$$



Example 2.04 (continued)

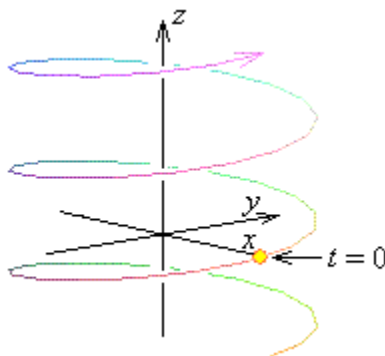
Find the tangential and normal components of velocity and acceleration for a particle travelling along the helix, given in parametric form by  $x = \cos t$ ,  $y = \sin t$ ,  $z = t$ .

Hence find the curvature. Assume SI units.

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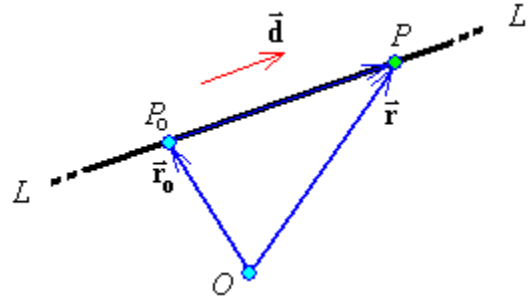
From page 2.09,

$$\bar{\mathbf{r}}(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix} \Rightarrow \bar{\mathbf{v}} = \frac{d\bar{\mathbf{r}}}{dt} = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix} \Rightarrow \bar{\mathbf{a}} = \frac{d\bar{\mathbf{v}}}{dt} = \begin{bmatrix} -\cos t \\ -\sin t \\ 0 \end{bmatrix}$$



**Review of Lines** [from MATH 2050]

Given a point  $P_0(x_0, y_0, z_0)$  known to be on a line  $L$ ,  
 a non-zero vector  $\vec{d} = [a \ b \ c]^T$  parallel to the line,  
 and vectors  $\vec{r} = [x \ y \ z]^T$  and  
 $\vec{r}_0 = \overrightarrow{OP_0} = [x_0 \ y_0 \ z_0]^T$ , the equations of the line  
 may be expressed in the vector form



$$\vec{r} = \vec{r}_0 + t \vec{d}, \quad (t \in \mathbb{R})$$

or, provided none of  $a, b, c$  are zero, in the symmetric Cartesian form

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

If, for example,  $b = 0$ , but  $a$  and  $c$  are non-zero, then the Cartesian form becomes

$$y = y_0, \quad \frac{x-x_0}{a} = \frac{z-z_0}{c}$$

Example 2.11

Find the Cartesian equation of the line through the points  $(1, 3, 5)$  and  $(4, -1, 7)$ .

**Review of Planes** [from MATH 2050]

A plane  $\Pi$  is defined by two vectors [ $\Pi$  is seen here edge-on  $\rightarrow$ ].

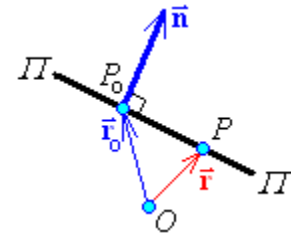
Its orientation is determined by a

non-zero normal vector  $\vec{n} = [A \ B \ C]^T$ .

Its location is determined by the position vector

$\vec{r}_o = \overrightarrow{OP_o} = [x_o \ y_o \ z_o]^T$  of any one point  $P_o$  known to be on the plane.

The vector  $\vec{r} = [x \ y \ z]^T$  is then the position vector of a point on the plane if and only if



$$\vec{r} \cdot \vec{n} = \vec{r}_o \cdot \vec{n}$$

or, equivalently,

$$Ax + By + Cz + D = 0$$

$$\text{where } D = -\vec{r}_o \cdot \vec{n} = -(Ax_o + By_o + Cz_o)$$

If  $\vec{u}$  and  $\vec{v}$  are two non-zero vectors not parallel to each other and both parallel to the plane, then a vector parametric form of the equation of the plane is

$$\vec{r} = \vec{r}_o + s\vec{u} + t\vec{v}, \quad (s, t \in \mathbb{R})$$

The normal vector to the plane is  $\vec{n} = \vec{u} \times \vec{v}$ , from which the other two forms above follow.

Example 2.12

Find the Cartesian equation of the plane through the points  $(0, 3, 5)$ ,  $(2, 4, 5)$  and  $(4, -1, 7)$  and find the equation of the normal line to that plane that passes through the origin.

Example 2.12 (continued)

[End of Chapter 2]

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