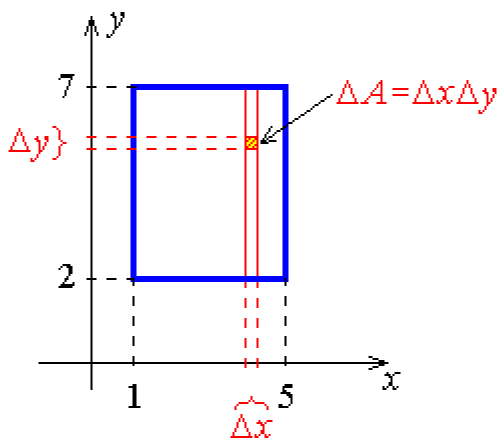
3. Multiple Integration

This chapter provides only a very brief introduction to the major topic of multiple integration. Uses of multiple integration include the evaluation of areas, volumes, masses, total charge on a surface and the location of a centre-of-mass. In future chapters we shall address the issue of integration over non-flat surfaces.

Double Integrals (Cartesian Coordinates)

Example 3.01

Find the area shown (assuming SI units).



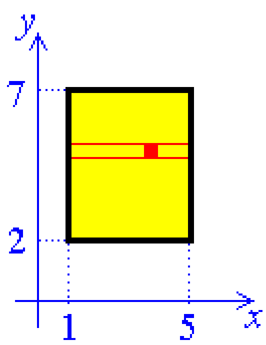
Example 3.01 (continued)

Suppose that the surface density on the rectangle is $\sigma = x^2 y$. Find the mass of the rectangle.

The element of mass is

OR

We can choose to sum horizontally first:



$$m = \int_2^7 \int_1^5 x^2 y \, dx \, dy$$

$$m = \int_2^7 y \left(\int_1^5 x^2 \, dx \right) dy$$

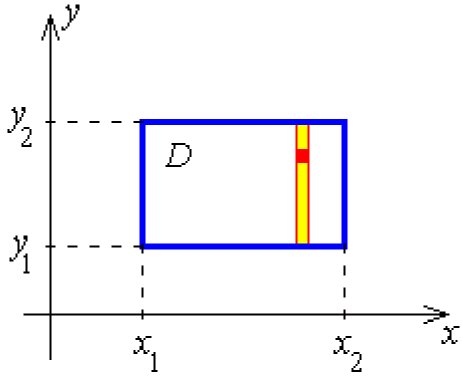
The inner integral has no dependency at all on y, in its limits or in its integrand. It can therefore be extracted as a "constant" factor from inside the outer integral.

$$m = \left(\int_1^5 x^2 \, dx \right) \cdot \left(\int_2^7 y \, dy \right)$$

which is exactly the same form as before, leading to the same value of 930 kg.

A double integral $\iint_D f(x,y)\,dA$ may be separated into a pair of single integrals if

- the region D is a rectangle, with sides parallel to the coordinate axes; and
- the integrand is separable: $f(x,y) = g(x) \cdot h(y)$.



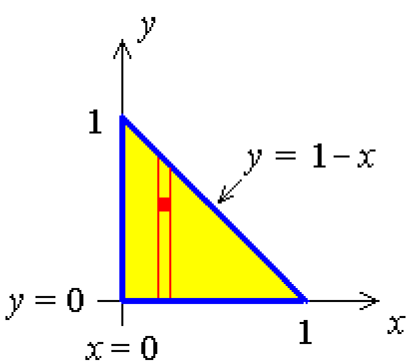
$$\iint_D f(x,y)\,dA = \int_{x_1}^{x_2} \int_{y_1}^{y_2} g(x)\,h(y)\,dy\,dx$$

$$= \left(\int_{y_1}^{y_2} h(y)\,dy \right) \cdot \left(\int_{x_1}^{x_2} g(x)\,dx \right)$$

This was the case in Example 3.01.

Example 3.02

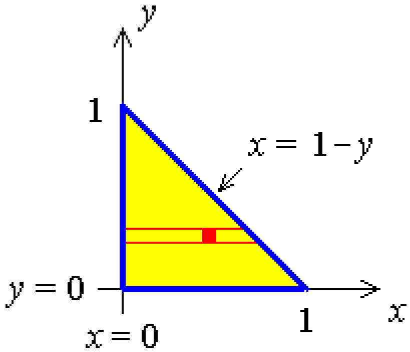
The triangular region shown here has surface density $\sigma = x + y$.
Find the mass of the triangular plate.



Example 3.02 (continued)

OR

We can choose to sum horizontally first (re-iterate):



Generally:

In Cartesian coordinates on the *xy*-plane, the rectangular element of area is

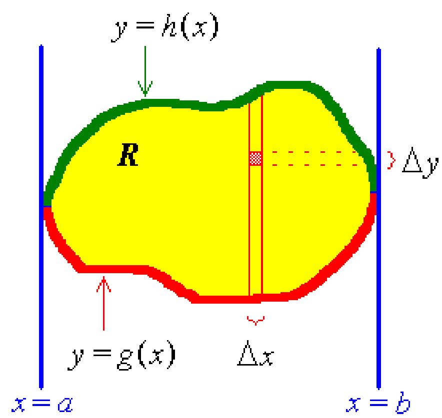
$$\Delta A = \Delta x \cdot \Delta y\,.$$

Summing all such elements of area along a vertical strip, the area of the elementary strip is

$$\left(\sum_{y=g(x)}^{h(x)} \Delta y \right) \Delta x$$

Summing all the strips across the region *R*, the total area of the region is:

$$A \;\approx\; \sum_{x=a}^{b} \left(\left(\sum_{y=g(x)}^{h(x)} \Delta y \right) \Delta x \right)$$



In the limit as the elements Δx and Δy shrink to zero, this sum becomes

$$A \;=\; \int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} 1\, dy\, dx$$

If the surface density σ within the region is a function of location, $\sigma = f\left(x, y\right)$, then the mass of the region is

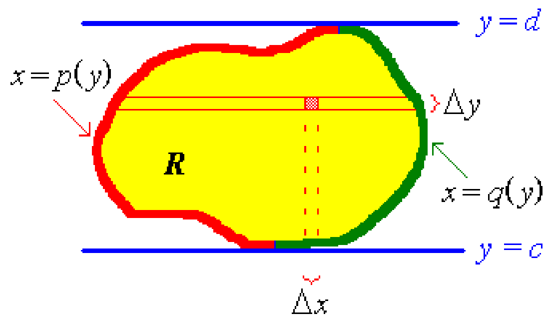
$$m \;=\; \int_{x=a}^{x=b} \left(\int_{y=g(x)}^{y=h(x)} f\left(x, y\right) dy \right) dx$$

The inner integral must be evaluated first.

Re-iteration:

One may reverse the order of integration by summing the elements of area ΔA horizontally first, then adding the strips across the region from bottom to top. This generates the double integral for the total area of the region

$$A = \int_{y=c}^{y=d} \left(\int_{x=p(y)}^{x=q(y)} 1\, dx \right) dy$$

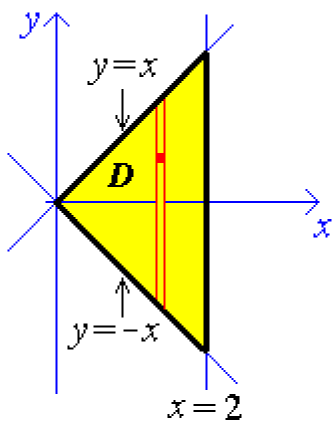


The mass becomes

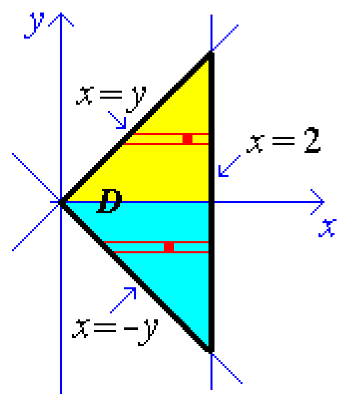
$$m = \int_{y=c}^{y=d} \left(\int_{x=p(y)}^{x=q(y)} f(x,y)\, dx \right) dy$$

Choose the orientation of elementary strips that generates the simpler double integration.

For example,



is preferable to



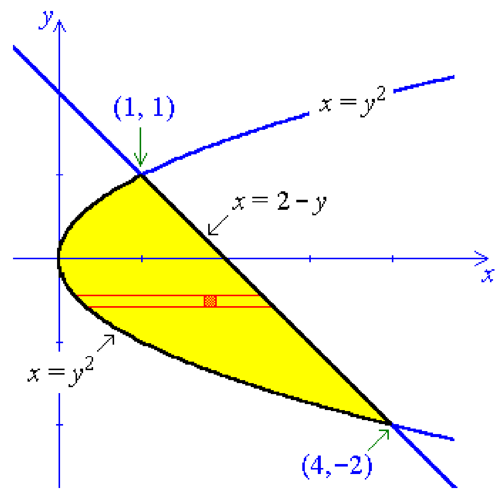
Example 3.03

Evaluate $I = \iint\limits_R \left(6x + 2y^2\right) dA$

where R is the region enclosed by the parabola $x = y^2$ and the line $x + y = 2$.

The upper boundary changes form at $x = 1$.
The left boundary is the same throughout R.
The right boundary is the same throughout R.
Therefore choose horizontal strips.

$$I = \int_{-2}^{1} \int_{y^2}^{2-y} \left(6x + 2y^2\right) dx\, dy$$



Polar Double Integrals

The Jacobian of the transformation from Cartesian to plane polar coordinates is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \left\| \begin{matrix} x_r & y_r \\ x_\theta & y_\theta \end{matrix} \right\| = r$$

The element of area is therefore

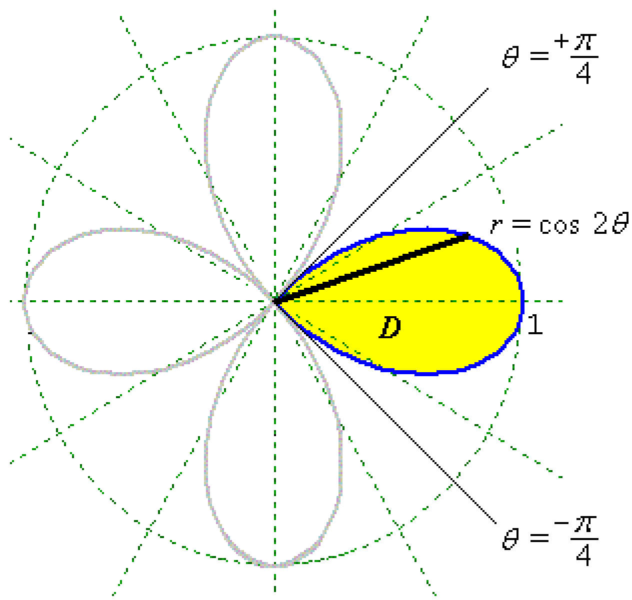
$$dA = dx\,dy = r\,dr\,d\theta$$

Example 3.04

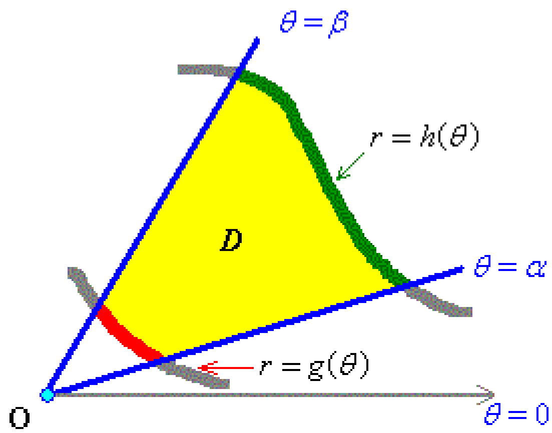
Find the area enclosed by one loop of the curve $r = \cos 2\theta$.

Boundaries:

Area:



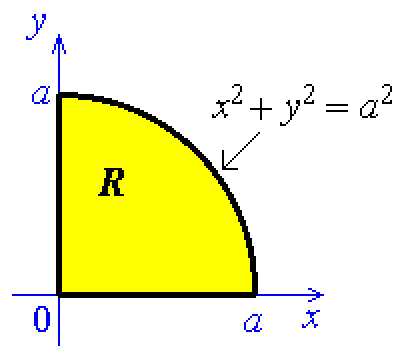
In general, in plane polar coordinates,



$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example 3.05

Find the centre of mass for a plate of surface density $\sigma = \dfrac{k}{\sqrt{x^2+y^2}}$, whose boundary is the portion of the circle $x^2+y^2=a^2$ that is inside the first quadrant. k and a are positive constants.



Use plane polar coordinates.

Boundaries:

The positive x-axis is the line $\theta = 0$.

The positive y-axis is the line $\theta = \pi/2$.

The circle is $r^2 = a^2$, which is $r = a$.

Mass:

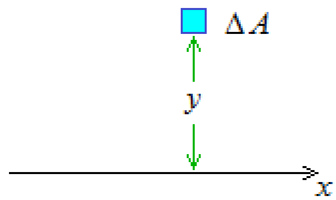
Surface density $\sigma = \dfrac{k}{\sqrt{x^2+y^2}}$.

Example 3.05 (continued)

First Moments about the *x*-axis:

Second Moments

Just as the element of first moment of mass about the *x*-axis is $\Delta M_x = y\,\Delta m = y\,\sigma\,\Delta A$,



the element of second moment of mass about the *x*-axis is $\Delta I_x = y^2\sigma\,\Delta A$ and
the element of second moment of mass about the *y*-axis is $\Delta I_y = x^2\sigma\,\Delta A$.

Summing all such elements over a plane region R in the limit as $\Delta A \to 0$, for a region R of surface density σ, the second moments of mass about the coordinate axes are

$$I_x = \iint_R y^2\sigma\,dA \quad \text{and} \quad I_y = \iint_R x^2\sigma\,dA$$

We are usually interested in the second moments about the centroid $(\bar{x}, \bar{y})$ (which is also the centre of mass when the density is constant):

$$I_x = \iint_R (y-\bar{y})^2\,\sigma\,dA \quad \text{and} \quad I_y = \iint_R (x-\bar{x})^2\,\sigma\,dA$$

Where possible, we choose coordinate axes that pass through the centre of mass.

We can also define the polar moment of mass (or moment of inertia)

$$I = I_x + I_y = \iint_R \left(x^2+y^2\right)\sigma\,dA = \iint_R r^2\sigma\,dA$$

This is the moment of inertia of the plane region about an axis, at right angles to the plane, passing through the centre of mass of the region.

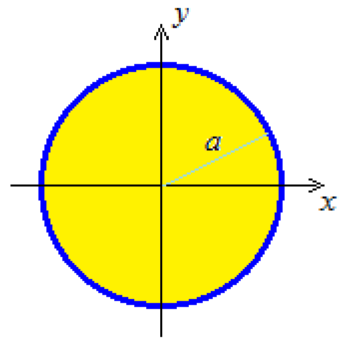
The SI units of moment of inertia are kg m^2.

The kinetic energy of a rigid body rotating at angular speed ω about its centre of mass is $E = \frac{1}{2}I\omega^2$.

Example 3.06

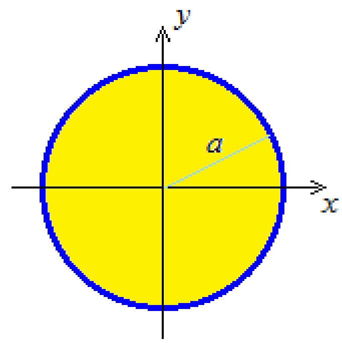
Find the moment of inertia I for a uniform circular disc of constant surface density σ and radius a about its centre of mass.

By symmetry, the centre of mass of a uniform circular disc is at the geometric centre of the disc. Place the origin there.
Use plane polar coordinates.



$$I_x = \iint_R y^2 \sigma \, dA =$$

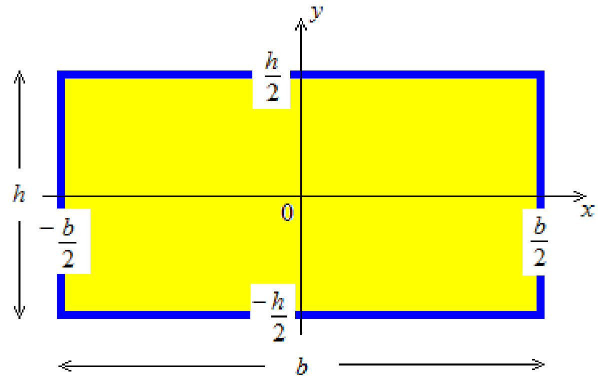
Example 3.06 (continued)



Example 3.07

Find the second moments of mass for a uniform rectangle of base b and height h about its centroid.

Place the origin at the centre of the rectangle.



Use Cartesian coordinates.
Note that $A = bh$ and $\sigma =$ constant.

$$I_x = \iint_R y^2 \sigma \, dA$$

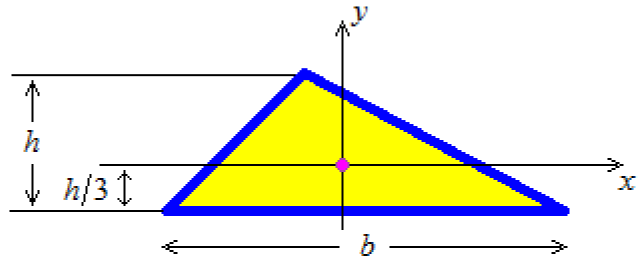
Example 3.07 (continued)

Two other standard second moments of area about the centroid for constant density are:

Triangle

$$A = \frac{bh}{2} \quad \Rightarrow \quad m = \frac{bh\sigma}{2}$$

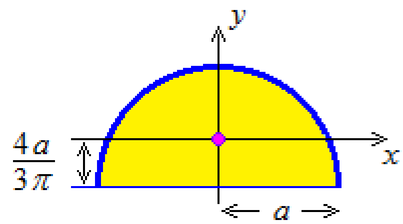
$$I_x = \frac{bh^3\sigma}{36} = \frac{mh^2}{18}$$



Semi-circle

$$A = \frac{1}{2}\pi a^2 \quad \Rightarrow \quad m = \frac{\sigma}{2}\pi a^2$$

$$I_x = \frac{\sigma\pi a^4}{8}\left(1 - \frac{64}{9\pi^2}\right) \approx .28 I_y \quad \text{and} \quad I_y = \frac{\sigma\pi a^4}{8}$$



The second moment of a collection of regions that share the same centroid is just the sum of the separate second moments.

When a region has a hole in it, centered on the centroid of the complete region, then the second moment of the region is the difference between the second moment for the complete region (with the hole filled in) and the second moment for the hole.

For example, the second moments of mass for an annulus of inner radius a and outer radius b are

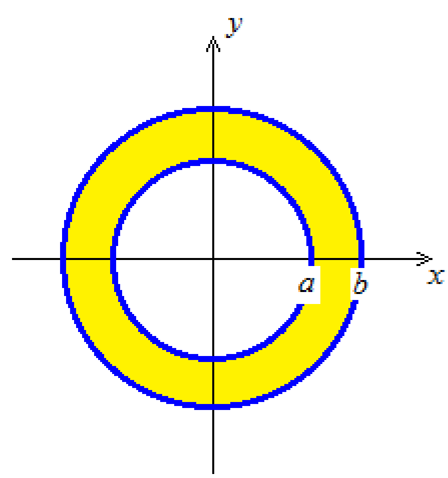
$$I_x = I_y = \frac{\sigma\pi\left(b^4 - a^4\right)}{4} \quad \text{and} \quad I = \frac{\sigma\pi\left(b^4 - a^4\right)}{2}$$

[using the results from example 3.06]

The mass of the annulus is $m = \sigma\pi\left(b^2 - a^2\right)$

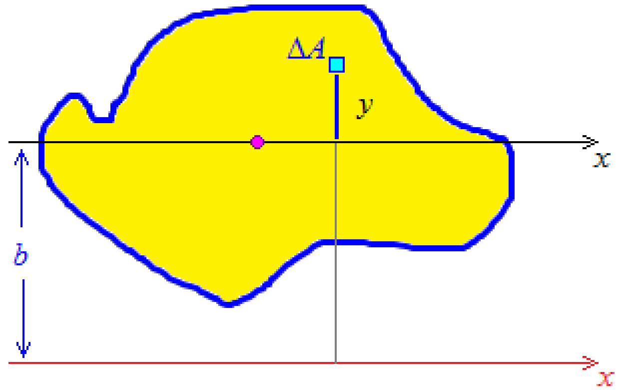
and $b^4 - a^4 = \left(b^2 - a^2\right)\left(b^2 + a^2\right)$, so that we can also write

$$I = \frac{m\left(b^2 + a^2\right)}{2}$$



Parallel Axis Theorem

The second moment of a composite shape can be found by shifting the reference axis of the standard second moment of each section from the centroidal axis of that section to a parallel axis that passes through the centroid of the composite shape.



If the x axis passes through the centroid then the second moment $I_{x'}$ about an axis x' parallel to the x axis and a distance b away from it will be related to I_x:

$$\Delta I_x = y^2 \Delta m = y^2 \sigma \, \Delta A \quad \Rightarrow \quad I_x = \iint_R y^2 \sigma \, dA$$

$$\Delta I_{x'} =$$

Two examples are in Problem Set 3.

Triple Integrals

The concepts for double integrals (surfaces) extend naturally to triple integrals (volumes). The element of volume, in terms of the Cartesian coordinate system (x, y, z) and another orthogonal coordinate system (u, v, w), is

$$dV = dx\,dy\,dz = \frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw$$

and

$$\iiint_V f(x,y,z)\,dV = \int_{w_1}^{w_2}\int_{v_1(w)}^{v_2(w)}\int_{u_1(v,w)}^{u_2(v,w)} f\big(x(u,v,w),\,y(u,v,w),\,z(u,v,w)\big)\frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw$$

The most common choices for non-Cartesian coordinate systems in $\mathbb{R}^3$ are:

Cylindrical Polar Coordinates:

$$x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

$$z = z$$

for which the differential volume is

$$dV = \frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}\,d\rho\,d\phi\,dz = \rho\,d\rho\,d\phi\,dz$$

Spherical Polar Coordinates:

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

for which the differential volume is

$$dV = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\,dr\,d\theta\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$

Example 3.08:

Verify the formula $V = \frac{4}{3}\pi a^3$ for the volume of a sphere of radius a.

Start by placing the origin at the centre of the sphere.

Example 3.09:

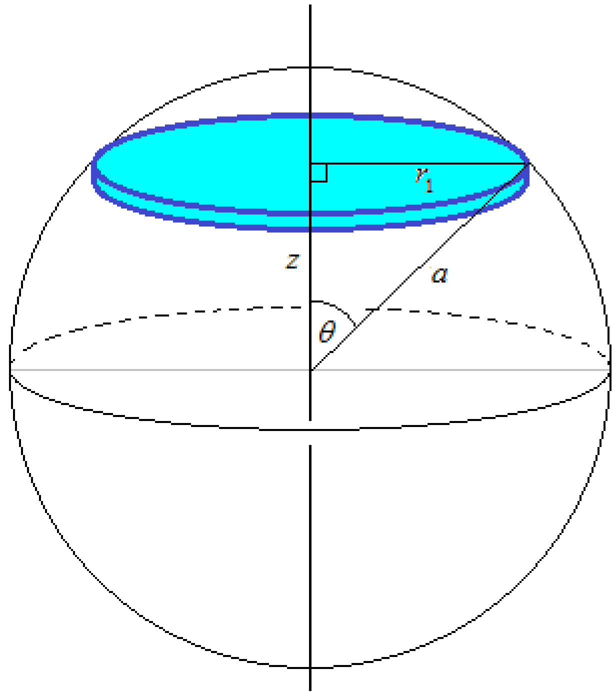
The density of an object is equal to the reciprocal of the distance from the origin.
Find the mass and the average density inside the sphere $r = a$.

Use spherical polar coordinates.
Density:

Mass:

Example 3.10:

Find the moment of inertia I for a region of uniform density ρ bounded by a sphere of radius a around any axis through the centre of the sphere.



Let the z axis be aligned with the rotation axis.

Any cross section through the region parallel to the equatorial plane is a circular disc, of radius

The mass of the disc is

$\Delta m \approx$

From example 3.06, the moment of inertia of this disc is

END OF CHAPTER 3