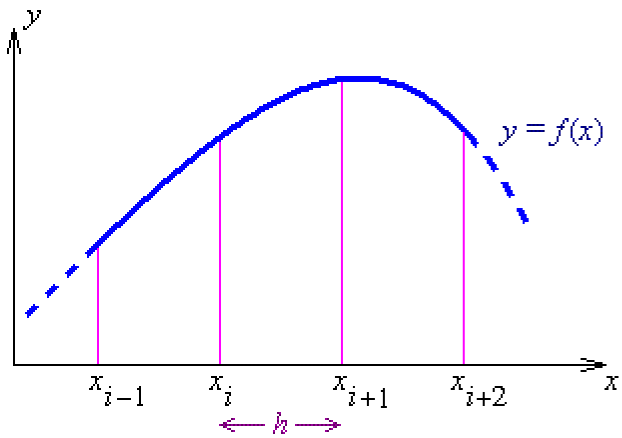
5. Numerical Integration

In some of our previous work, (most notably the evaluation of arc length), it has been difficult or impossible to find the indefinite integral. Various symbolic algebra and calculus software packages (such as Maple®) may be able to provide either an exact answer or a numerical approximation.

In this brief chapter we shall see three numerical schemes for the evaluation of proper definite integrals and two methods for finding the zeroes of functions.

In your first course in integral calculus, the definite integral $\int_a^b f(x)\,dx$ was constructed as a limit of the sum of the areas of rectangles fitted under the curve $y = f(x)$:



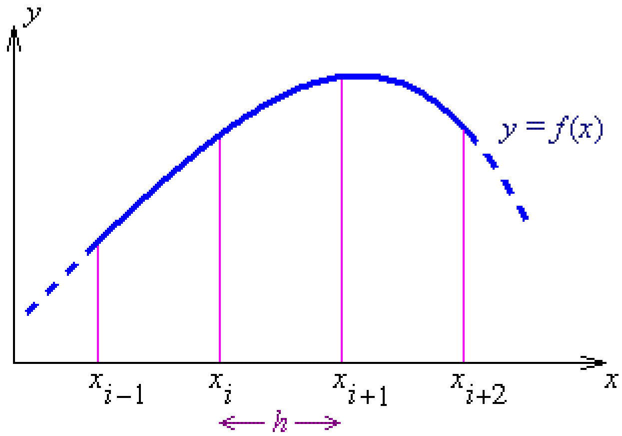
For n rectangles of equal width $h = \dfrac{b-a}{n}$, $x_i = a + ih \quad (i = 0, 1, 2, \ldots, n)$

(so that $x_0 = a$ and $x_n = b$) an approximation is

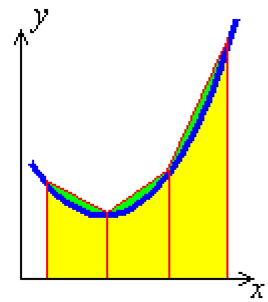
$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n-1} h \cdot f(x_i)$$

However, unless h is very small (which requires a very large number of rectangles), it is clear that this approximation can lead to substantial errors.

A better approximation is to join the top left corner of each rectangle to the top left corner of the next rectangle, thus replacing the rectangles by trapezoids:



Where a curve $y = f(x)$ is concave down, the area of a trapezoid will be an underestimate of the area under the curve. Where the curve is concave up, the area of the trapezoid will be an overestimate of the area under the curve. The sum of the areas of the trapezoids usually provides a better estimate of the area under the curve than the sum of the areas of the rectangles does.



The area of the trapezoid with left edge at $x = x_i$ is $A_i =$

Abbreviate $f(x_i)$ by f_i. It follows that

$$\int_a^b f(x)\,dx \approx$$

Example 5.01

Estimate the length along the parabola $y = x^2$ from $x = 0$ to $x = 2$, using the trapezoidal rule with $n = 8$.

$$y = x^2 \qquad \Rightarrow \quad \frac{dy}{dx} = 2x$$

Table of values for the trapezoidal rule:

i	x_i	$f(x_i)$	$2f(x_i)$
0	0.00	1.000000	
1	0.25	1.118034	2.236068
2	0.50	1.414214	2.828427
3	0.75	1.802776	3.605551
4	1.00	2.236068	4.472136
5	1.25	2.692582	5.385165
6	1.50	3.162278	6.324555
7	1.75	3.640055	7.280110
8	2.00	4.123106	

Therefore $\int_0^2 f(x)\,dx \approx 4.657$

Example 5.01 (continued)

Note: after more than one substitution, it can be shown that

$$\int_a^b \sqrt{1+4x^2}\,dx = \left[\frac{2x\sqrt{1+4x^2} + \sinh^{-1}(2x)}{4}\right]_a^b = \left[\frac{2x\sqrt{1+4x^2} + \ln\left(2x+\sqrt{1+4x^2}\right)}{4}\right]_a^b$$

Therefore the exact value is $L = \sqrt{17} + \dfrac{\sinh^{-1}(4)}{4} \approx 4.647$

The accuracy of the trapezoidal rule does improve with larger numbers of narrower intervals, but at the cost of more computations.

In this example, the integrand is concave up everywhere. Therefore the trapezoidal rule will provide an overestimate that is worse for smaller *n*.

n	*L*
8	4.6569
16	4.6493
32	4.6474
64	4.6469
Exact	**4.6468**

Also see the Excel file at "`www.engr.mun.ca/~ggeorge/4430/demos/`".

The trapezoidal rule essentially estimates the curve by straight lines between pairs of adjacent points. A further refinement involves fitting a parabola through each set of three consecutive points. The resulting algorithm is **Simpson's Rule**:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \ldots + 2f_{n-2} + 4f_{n-1} + f_n\right)$$

where the number n of intervals must be even.

This algorithm is attributed to the eighteenth century British mathematician Thomas Simpson, but it was discovered a century earlier by astronomer Johannes Kepler.

Example 5.02

Estimate the length along the parabola $y = x^2$ from $x = 0$ to $x = 2$, using Simpson's rule with $n = 4$.

As in Example 5.01, the arc length is $L = \int_0^2 \sqrt{1+4x^2}\, dx$.

i	x_i	$f(x_i)$	$2f(x_i)$	$4f(x_i)$
0	0.00	**1.000000**		
1	0.50	1.414214		**5.656854**
2	1.00	2.236068	**4.472136**	
3	1.50	3.162278		**12.64911**
4	2.00	**4.123106**		

$$L = \int_0^2 \sqrt{1+4x^2}\, dx \approx \frac{1}{3}\cdot\frac{1}{2}\left(1.000000+5.656854+4.472136+12.64911+4.123106\right)$$

$\Rightarrow \quad L \approx 4.650$

which is quite close to the exact answer of 4.647 (to 3 d.p.), given the small number of intervals used. It is a better approximation than the trapezoidal rule with twice as many intervals.

Graphical Solution to $f(x) = 0$

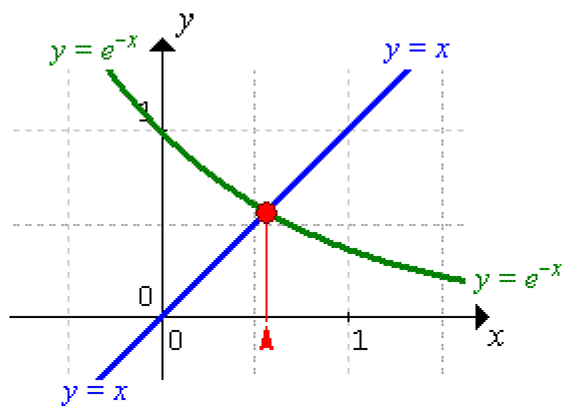
In cases where it is difficult to find a zero of a function (that is, values of x for which $f(x) = 0$) analytically, various numerical schemes exist.

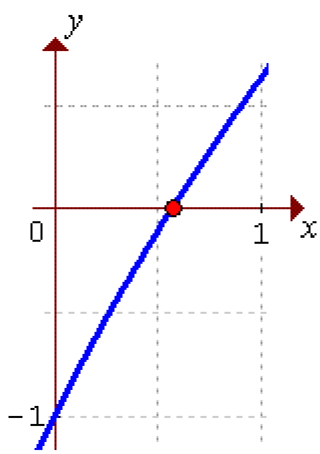
The simplest method is graphical. Using appropriate software (even a simple handheld graphing calculator), just zoom in on the x-axis intercept repeatedly until the desired precision is achieved.

Example 5.03

Find the solution of $e^{-x} = x$, correct to five decimal places.

From a sketch of the two curves $y = x$ and $y = e^{-x}$, it is obvious that the only solution is somewhere in the interval (0, 1).





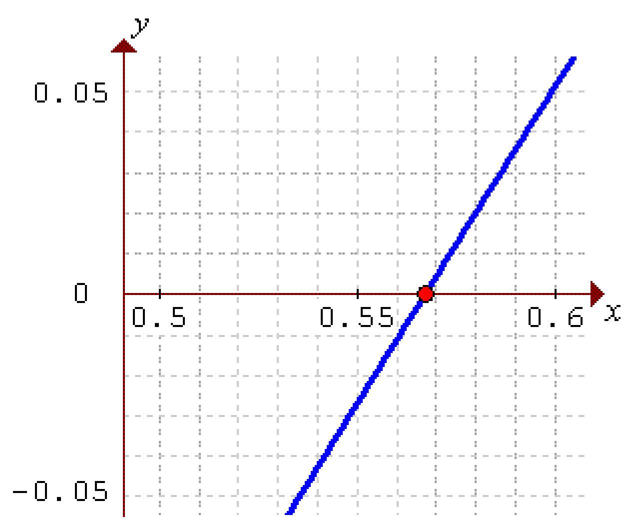
Graph the curve $y = f(x) = x - e^{-x}$.

Clearly $f(0) = -1 < 0$
and $f(1) = 1 - 1/e > 0$

$f(x)$ is continuous and changes sign only once inside (0, 1).

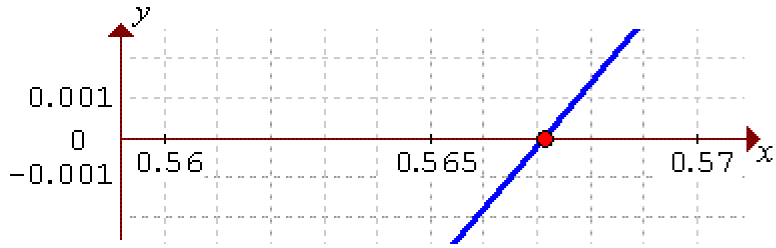
Zoom in to the interval $x = 0.5$ to 0.6:

Clearly the root of $f(x) = 0$ is in (0.565, 0.570).

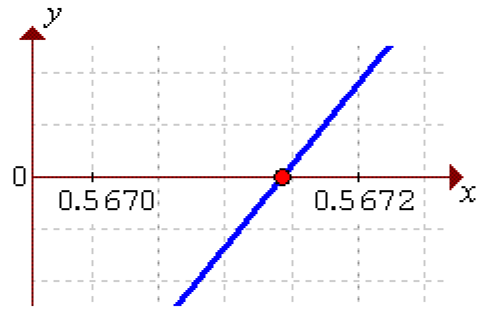


Example 5.03 (continued)

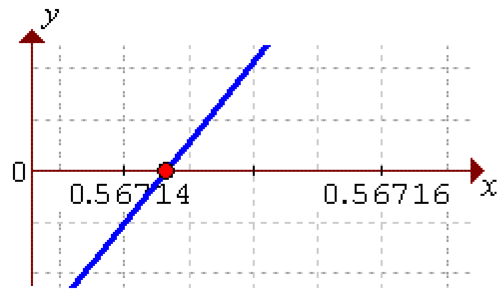
Zooming in again,



Now the root is seen to be in (0.5670, 0.5673).



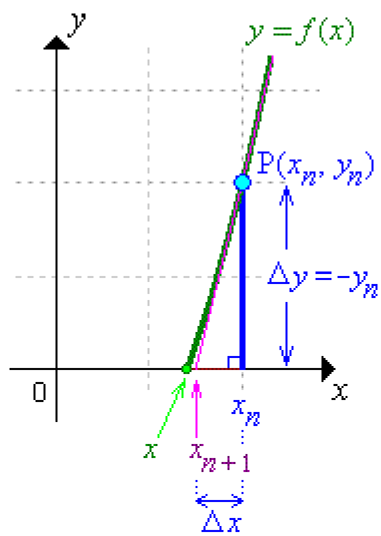
The root seems to be in (0.56713, 0.56715). One final zoom will resolve the fifth decimal place.



Therefore, correct to five decimal places, the solution to $e^{-x} = x$ is $x = 0.56714$.
A calculator quickly confirms that $e^{-0.56714} \approx 0.56714$.

Numerical Solution to $f(x) = 0$

Newton's Method



From the definition of the derivative, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$,

we obtain $\Delta y \approx \frac{dy}{dx}\Delta x$ or, equivalently,

$$\Delta x \approx \frac{\Delta y}{f'(x)}.$$

The tangent line to the curve $y = f(x)$ at the point $\mathrm{P}(x_n, y_n)$ has slope $= f'(x_n)$.

Follow the tangent line down to its x axis intercept. That intercept is the next approximation x_{n+1}.

$$\Delta y = y_{n+1} - y_n = 0 - y_n = -f(x_n) \text{ and}$$

$$\Delta x = x_{n+1} - x_n$$

$$\Rightarrow \quad x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$$

If x_n is the n^{th} approximation to the equation $f(x) = 0$, then a better approximation may be

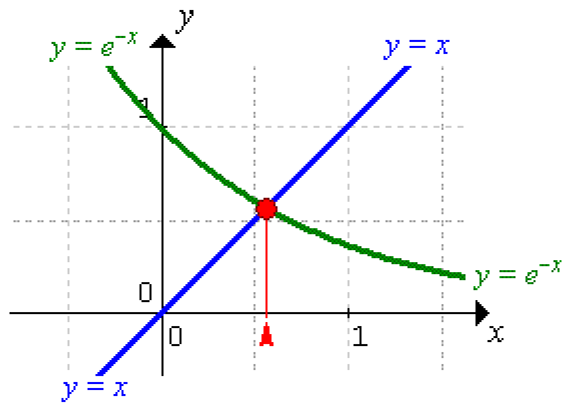
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

which is Newton's method, (first developed in a work of Sir Isaac Newton, written in 1669 but not published until 1711).

Example 5.04

Find the solution of $x = e^{-x}$, correct to 5 decimal places.

From a sketch of the two curves $y = x$ and $y = e^{-x}$, it is obvious that the only solution is somewhere in the interval (0, 1). A reasonable first guess is $x_0 = \frac{1}{2}$.



$$f(x) = x - e^{-x} \quad \Rightarrow \quad f'(x) = 1 + e^{-x}$$

$$\Rightarrow \quad x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}.$$

Table of consecutive values:

x_n	$f(x_n) = x_n - e^{-x_n}$	$f'(x_n) = 1 + e^{-x_n}$	$\dfrac{f(x_n)}{f'(x_n)}$
0.500000	–0.106531	1.606531	–0.066311
0.566311	–0.001305	1.567616	–0.000832
0.567143	0.000000	1.567143	0.000000
0.567143			

Correct to five decimal places, the solution to $x = e^{-x}$ is **x = 0.56714** .
In fact, we have the root correct to six decimal places, x = 0.567143.

A spreadsheet to demonstrate Newton's method for this example is available from the course web site, at "`www.engr.mun.ca/~ggeorge/4430/demos/`".

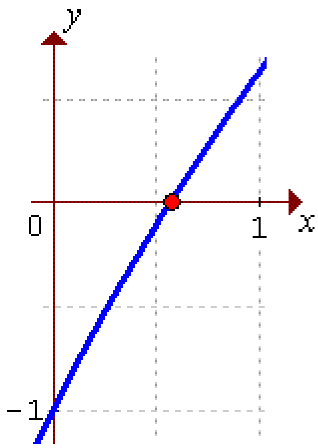
This method converges more rapidly than the graphical method, but requires more computational effort.

Caution:
Newton's method can fail if $f'(x) = 0$ in the neighbourhood of the root. A shallow tangent line could result in a sequence of approximations that fails to converge to the correct value. This method also should not be used near any discontinuities in $f(x)$.

There is a wealth of other methods for numerical integration and for the numerical solution of various equations, which we do not have the time to explore in this course.

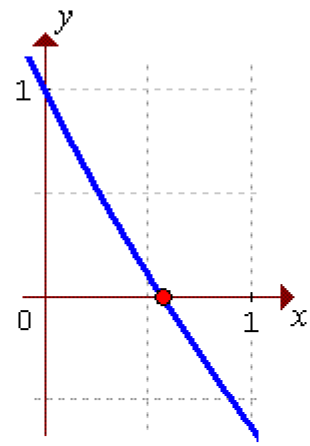
Example 5.04 (additional note)

Graph of $f(x) = x - e^{-x}$,



or, if $y = x$ and $y = e^{-x}$ are taken the other way around,

Graph of $f(x) = e^{-x} - x$



Reversing the order of the two functions reverses the signs of all entries in the table for the second and third columns, $f(x_n)$ and $f'(x_n)$, but the entries in the first (x_n) and last $\left(\frac{f(x_n)}{f'(x_n)}\right)$ columns will be exactly the same.

[End of Chapter 5]