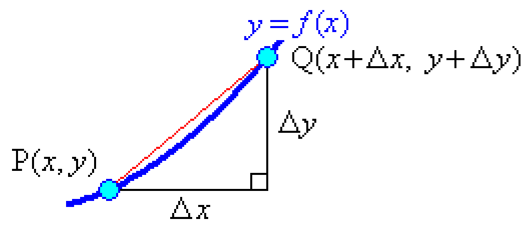
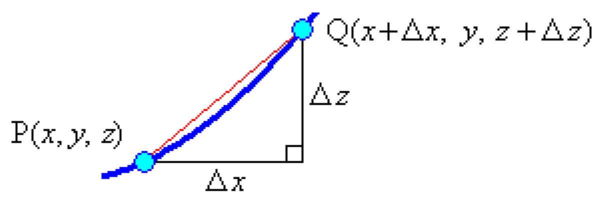
6. The Gradient Vector - Review

If a curve in $\mathbb{R}^2$ is represented by $y = f(x)$, then



$$\frac{dy}{dx} = \lim_{Q \to P} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If a surface in $\mathbb{R}^3$ is represented by $z = f(x, y)$, then in a slice y = constant,

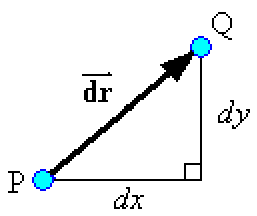


$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z+\Delta z) - f(x, y, z)}{\Delta x}$$

Similarly,

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z+\Delta z) - f(x, y, z)}{\Delta y}$$

In the plane of the independent variables:



$$f(\text{P}) = f$$
$$f(\text{Q}) = f + df$$
$$\overrightarrow{\mathbf{dr}} = \begin{bmatrix} dx & dy & 0 \end{bmatrix}^T$$

Chain rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\Rightarrow \quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\Rightarrow \quad df = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{bmatrix}^{\mathrm{T}} \bullet \begin{bmatrix} dx & dy \end{bmatrix}^{\mathrm{T}} = \vec{\nabla} f \bullet d\vec{\mathbf{r}}$$

where $\vec{\nabla} f$ (pronounced as "del f") is the **gradient vector**.

At any point (x, y) in the domain, the value of the function $f(x, y)$ changes at different rates when one moves in different directions on the xy-plane.

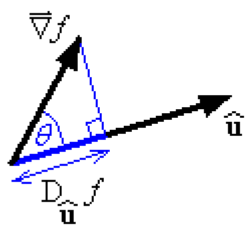
$\vec{\nabla} f$ is a vector in the plane of the independent variables (the xy-plane).
The magnitude of $\vec{\nabla} f$ at a point (x, y) is the maximum instantaneous rate of increase of f at that point. The direction of $\vec{\nabla} f$ at that point is the direction in which one would have to start moving on the xy-plane in order to experience that maximum rate of increase, (which is also at right angles to the contour $f(x, y) = \text{constant}$ at that point).

Points where $\vec{\nabla} f = \vec{\mathbf{0}}$ are critical points of f, (maximum, minimum or saddle point).

The **directional derivative** of f in the direction of the unit vector $\hat{\mathbf{u}}$ is

$$\boxed{D_{\hat{\mathbf{u}}} f = \vec{\nabla} f \bullet \hat{\mathbf{u}}}$$

Both vectors are in the plane of the independent variables.
The directional derivative is the component of $\vec{\nabla} f$ in the direction of $\hat{\mathbf{u}}$.



$$D_{\hat{\mathbf{u}}} f = \left|\vec{\nabla} f\right| \left|\hat{\mathbf{u}}\right| \cos\theta$$

The results above can be extended to functions of more than two variables.
For the hypersurface $z = f\left(x_1, x_2, \ldots, x_n\right)$ in $\mathbb{R}^{n+1}$, the chain rule becomes

$$\frac{df}{dt} = \vec{\nabla} f \bullet \frac{d\vec{\mathbf{r}}}{dt}, \text{ where}$$

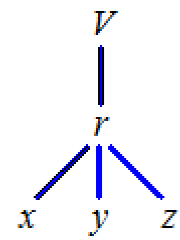
$$\vec{\nabla} f = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \cdots & \dfrac{\partial f}{\partial x_n} \end{bmatrix}^{\mathrm{T}} \quad \text{and} \quad \frac{d\vec{\mathbf{r}}}{dt} = \begin{bmatrix} \dfrac{dx_1}{dt} & \dfrac{dx_2}{dt} & \cdots & \dfrac{dx_n}{dt} \end{bmatrix}^{\mathrm{T}}$$

Example 6.01

The electrostatic potential V at a point $P(x, y, z)$ in $\mathbb{R}^3$ due to a point charge Q at the origin is

$$V = \frac{1}{4\pi\varepsilon} \cdot \frac{Q}{r}, \quad \text{where} \quad r = \sqrt{x^2 + y^2 + z^2}\,.$$

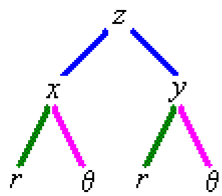
Find the rate of change of V at the point (1, 2, 2) in the direction $\hat{\mathbf{k}} - 2\hat{\mathbf{i}}$.
Find the maximum value of the directional derivative over all directions at any point.
Find the level surfaces.



Example 6.01 (continued)

Change of coordinates:

Suppose $z = f(x, y)$ (where (x, y) are Cartesian coordinates) and $\dfrac{\partial z}{\partial r}$ is wanted, (where (r, θ) are plane polar coordinates). Then



$$\Rightarrow \quad \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

But $\quad x = r\cos\theta, \quad y = r\sin\theta$

$$\Rightarrow \quad \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}(r\cos\theta)$ can be found in a similar way.

In matrix form, the chain rule can be expressed concisely as

$$\begin{bmatrix} \dfrac{\partial z}{\partial r} \\[2ex] \dfrac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\[2ex] \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial z}{\partial x} \\[2ex] \dfrac{\partial z}{\partial y} \end{bmatrix} .$$

Note that

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \operatorname{abs}\left(\det \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\[2ex] \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} \right)$$ is the **Jacobian**

For the transformation from Cartesian to plane polar coordinates in $\mathbb{R}^2$, the Jacobian is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \left\| \begin{matrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{matrix} \right\| = \left| r\cos^2\theta + r\sin^2\theta \right| = r$$

Integrals over areas can therefore be transformed using the Jacobian:

$$\iint_A f(x, y)\, dx\, dy = \iint_A f(x, y) \frac{\partial(x, y)}{\partial(r, \theta)}\, dr\, d\theta = \iint_A g(r, \theta)\, r\, dr\, d\theta$$

where $f(x, y) = g(r, \theta)$ at all points in the area A of integration.
We shall return to this topic later.

Surfaces

The general Cartesian equation of a surface in $\mathbb{R}^3$ (whether a plane or not) is of the form

$$f(x, y, z) = c$$

Imposing one constraint in a three-dimensional volume removes one degree of freedom, leaving a two-dimensional surface.

At every point on the surface where $\vec{\nabla} f$ exists as a non-zero vector, $\vec{\nabla} f$ is orthogonal (perpendicular) to the level surface of the function f that passes through that point. Therefore, at every point on the surface $f(x, y, z) = c$,

the gradient vector $\vec{\nabla} f$ is normal to the tangent plane.

The **tangent plane** at the point $P(x_o, y_o, z_o)$ to the surface $f(x, y, z) = c$ has the equation

$$\boxed{\vec{\mathbf{n}} \bullet \vec{\mathbf{r}} = \vec{\mathbf{n}} \bullet \vec{\mathbf{r}}_o, \quad \text{where} \quad \vec{\mathbf{n}} = \vec{\nabla} f \big|_P}$$

This formula fails only at locations where $\vec{\nabla} f = \vec{\mathbf{0}}$.

Let $\vec{\nabla} f \big|_P = [\, n_1 \;\; n_2 \;\; n_3 \,]^T$, then the **normal line** at the point $P(x_o, y_o, z_o)$ to the surface $f(x, y, z) = c$ has the equations

$$\boxed{\frac{x - x_o}{n_1} = \frac{y - y_o}{n_2} = \frac{z - z_o}{n_3}}$$

(which must be modified if any of the components n_1, n_2, n_3 is zero).

Example 6.02

Find the Cartesian equations of the tangent plane and normal line to the surface $z = x^2 + y$ at the point $(-1, 1, 2)$.

Example 6.03

Find the angle between the surfaces $x^2 + y^2 + z^2 = 4$ and $z^2 + x^2 = 2$ at the point $(1, \sqrt{2}, 1)$.

Note: In the event that $\vec{\mathbf{n}}_1 \bullet \vec{\mathbf{n}}_2 < 0$, then the two normal vectors meet at an obtuse angle (they are pointing in approximately opposite directions). In that case use $\left| \vec{\mathbf{n}}_1 \bullet \vec{\mathbf{n}}_2 \right|$ to obtain the acute angle.

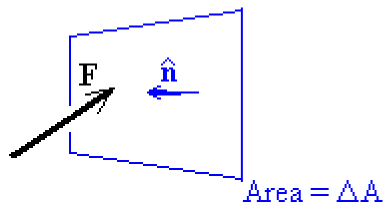
Gradient Operator

For three independent variables (x, y, z), the gradient operator is the "vector"

$$\vec{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}^{\mathrm{T}}$$

It operates on anything immediately to its right; either a scalar function or scalar field, or, via a dot product or cross product, on a vector function or vector field.

Divergence



For an elementary area ΔA in a vector field $\mathbf{F}$,

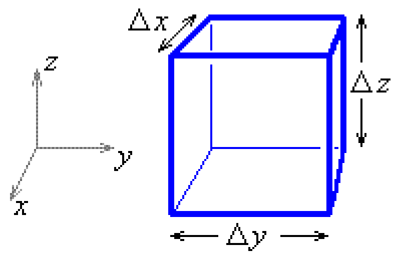
$\hat{\mathbf{n}}$ is an outward unit normal to the surface.

ΔA is sufficiently small that $\vec{\mathbf{F}} = [\, f_1 \;\; f_2 \;\; f_3 \,]^{\mathrm{T}}$ is approximately constant over ΔA.

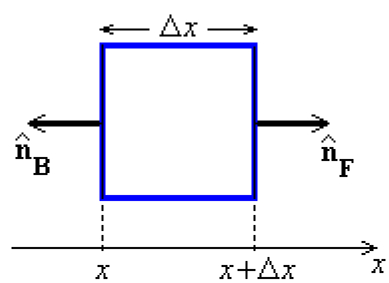
The element of flux $\Delta\phi$ from the vector field through ΔA is

$$\Delta\phi \approx \vec{\mathbf{F}} \bullet \hat{\mathbf{n}}\, \Delta A$$

Now add up the elements of flux passing through the six faces of an elementary cuboid of sides $\Delta x, \Delta y, \Delta z$, volume $\Delta V = \Delta x\, \Delta y\, \Delta z$ and with one corner at (x, y, z).



The front face is at ($x + \Delta x$) and the back face is at (x).



Back face: $\hat{\mathbf{n}}_{\mathbf{B}} =$

$\Delta A =$

$\Rightarrow \quad \vec{\mathbf{F}} \bullet \hat{\mathbf{n}}_{\mathbf{B}} =$

$\Rightarrow \quad \Delta\phi_B =$

Divergence (continued)

Front face: $\hat{\mathbf{n}}_{\mathbf{F}} =$

$\Delta A =$

$\Rightarrow \quad \vec{\mathbf{F}} \bullet \hat{\mathbf{n}}_{\mathbf{F}} =$

$\Rightarrow \quad \Delta\phi_F =$

$$\Rightarrow \quad \frac{\Delta\phi_{\mathrm{F}} + \Delta\phi_{\mathrm{B}}}{\Delta V} =$$

<u>Example 6.04</u>

Find the divergence of the vector **F**, given that $\mathbf{F} = -\nabla\phi$, $\phi = \dfrac{Q}{4\pi\varepsilon r}$, $r = \sqrt{x^2 + y^2 + z^2}$.

From example 6.01,

$$\vec{\nabla}\phi = \frac{-Q}{4\pi\varepsilon\, r^3}\vec{\mathbf{r}} \quad \Rightarrow \quad \vec{\mathbf{F}} = -\vec{\nabla}\phi = \frac{+Q}{4\pi\varepsilon\, r^3}\vec{\mathbf{r}}$$

Streamlines for Fluid Flow

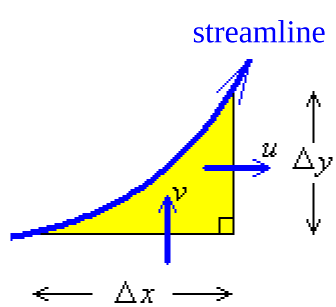
Let $\vec{\mathbf{v}}(\vec{\mathbf{r}})$ be the velocity at any point (x, y, z) in an incompressible fluid. Because the fluid is incompressible, the flow in to any region must be matched by the flow out from that region (except when the region includes a source or a sink). This generates the **continuity equation**

$$\text{div } \mathbf{v} = 0$$

Let us take the case of fluid flow parallel to the *x*-*y* plane everywhere, so that we can ignore the third dimension and consider the flow in two dimensions only. Then

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}(x, y) = u(x, y)\hat{\mathbf{i}} + v(x, y)\hat{\mathbf{j}}$$

The continuity equation then becomes



<u>Example 6.05</u> (Example 4.03 repeated)

Find the streamlines associated with the velocity field $\vec{\mathbf{v}} = \left[\dfrac{-y}{x^2+y^2} \quad \dfrac{x}{x^2+y^2} \right]^{\mathrm{T}}$ and find the streamline through the point (1, 0).

$$u = \frac{-y}{x^2+y^2}, \quad v = \frac{x}{x^2+y^2}$$

Verify that the equation of continuity is satisfied:

Divergence (a scalar quantity):

$$\boxed{\operatorname{div} \vec{\mathbf{F}} = \vec{\nabla} \bullet \vec{\mathbf{F}}}$$

Curl (a vector quantity):

$$\boxed{\operatorname{curl} \vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{F}} = \det \begin{pmatrix} \hat{\mathbf{i}} & \dfrac{\partial}{\partial x} & F_1 \\ \hat{\mathbf{j}} & \dfrac{\partial}{\partial y} & F_2 \\ \hat{\mathbf{k}} & \dfrac{\partial}{\partial z} & F_3 \end{pmatrix} = \begin{bmatrix} \dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z} \\ \dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x} \\ \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} \end{bmatrix}}$$

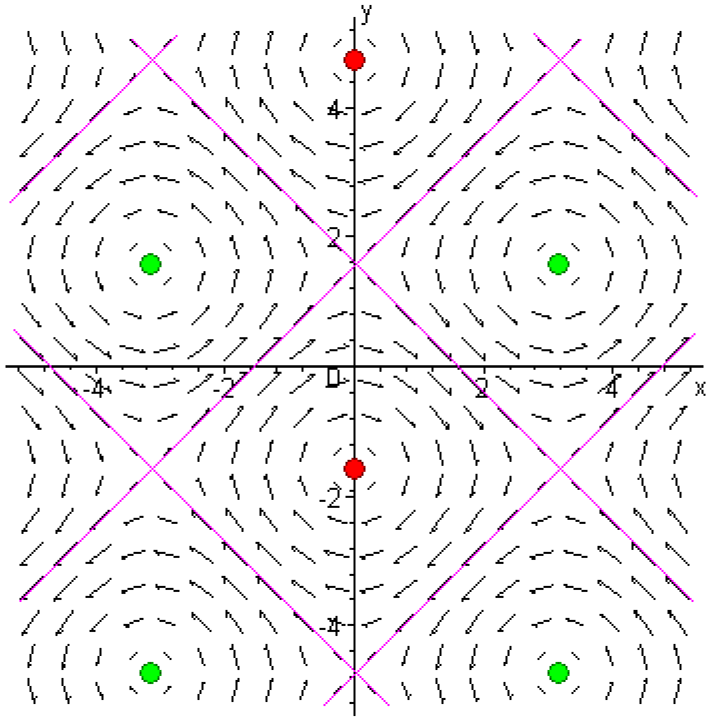
curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$ everywhere $\Rightarrow$ $\vec{\mathbf{F}}$ is an **irrotational vector field**.
Imagine a test particle immersed in and moving with a fluid. If the fluid motion does not cause the particle to rotate on its own axis (even if the particle is swept along a curved path), then the fluid flow is irrotational.

Example 6.06

Find curl $\vec{\mathbf{F}}$ for $\vec{\mathbf{F}} = \left[\cos y \;\; -\sin x \;\; 0\right]^{\mathrm{T}}$.
Also find the lines of force for the vector field $\vec{\mathbf{F}}$.

Example 6.06 (continued)

Direction field plot for the vector field $\mathbf{F} = [\cos y \ \ -\sin x \ \ 0]^{T}$:



$$\operatorname{curl} \vec{\mathbf{F}} = (\sin y - \cos x)\,\hat{\mathbf{k}}$$

Lines of force: $\sin y = A + \cos x \quad (-2 \le A \le 2)$

and $\operatorname{curl} \vec{\mathbf{F}} = A\,\hat{\mathbf{k}}$.

Along the highlighted lines (at 45° angles), $A = 0$ (and therefore curl $\mathbf{F} = \mathbf{0}$).
Where those lines cross, $\mathbf{F} = \mathbf{0}$ also, (in addition to $A = 0$ and curl $\mathbf{F} = \mathbf{0}$).
Half way along the lines between those intersections, | $\mathbf{F}$ | is at a maximum ($\sqrt{2}$).
At the highlighted dots, | curl $\mathbf{F}$ | achieves its maximum value of 2 and $\mathbf{F} = \mathbf{0}$ and where the lines of force near a dot are in an anticlockwise direction, curl $\mathbf{F} = +2\,\mathbf{k}$; where the lines of force near a dot are in a clockwise direction, curl $\mathbf{F} = -2\,\mathbf{k}$.

All differentiable gradient-vector fields are irrotational:

$$\boxed{\text{curl grad}\,\phi \;\equiv\; \vec{\nabla}\times\vec{\nabla}\phi \;\equiv\; \vec{\mathbf{0}}}$$

Proof:

Also:

$$\boxed{\text{div curl}\,\vec{\mathbf{F}} \;\equiv\; \vec{\nabla}\bullet\vec{\nabla}\times\vec{\mathbf{F}} \;\equiv\; 0}$$

Proof:

Let $\vec{\mathbf{F}} = [\, f_1 \;\; f_2 \;\; f_3 \,]^{\mathrm{T}}$ and $f_{1x} = \dfrac{\partial f_1}{\partial x}$ etc., then

The **Laplacian** of a twice-differentiable scalar field ϕ is:

$$\text{div grad}\,\phi \equiv \begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{bmatrix}^{\mathrm{T}} \bullet \begin{bmatrix} \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} & \dfrac{\partial \phi}{\partial z} \end{bmatrix}^{\mathrm{T}} \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \equiv \nabla^2 \phi$$

Laplace’s equation is

$$\nabla^2 \phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Some Vector Identities

$$\vec{\nabla} \times \vec{\nabla} f \;=\; \text{curl grad } f \;=\; \vec{\mathbf{0}}$$
$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{\mathbf{F}} \;=\; \text{div curl } \vec{\mathbf{F}} \;=\; 0$$

$$\text{Laplacian of } V \;=\; \nabla^2 V \;=\; \vec{\nabla} \cdot \left(\vec{\nabla} V \right) \;=\; \text{div grad } V$$

$$\vec{\nabla}(f\,g) \;=\; f\,\vec{\nabla} g \;+\; \left(\vec{\nabla} f\right) g$$

$$\vec{\nabla}^2 \vec{\mathbf{F}} \;=\; \vec{\nabla}\left(\vec{\nabla} \cdot \vec{\mathbf{F}}\right) \;-\; \vec{\nabla} \times \left(\vec{\nabla} \times \vec{\mathbf{F}}\right) \;=\; \text{grad div } \vec{\mathbf{F}} \;-\; \text{curl curl } \vec{\mathbf{F}}$$

$$\vec{\nabla} \cdot \left(g\,\vec{\mathbf{F}}\right) \;=\; \left(\vec{\nabla} g\right) \cdot \vec{\mathbf{F}} \;+\; g\left(\vec{\nabla} \cdot \vec{\mathbf{F}}\right)$$
div (g **F**) = (grad g)•**F** + g div **F**

$$\vec{\nabla} \times \left(g\,\vec{\mathbf{F}}\right) \;=\; \left(\vec{\nabla} g\right) \times \vec{\mathbf{F}} \;+\; g\left(\vec{\nabla} \times \vec{\mathbf{F}}\right)$$
curl (g **F**) = (grad g)×**F** + g curl **F**

$$\vec{\nabla} \cdot \left(\vec{\mathbf{F}} \times \vec{\mathbf{G}}\right) \;=\; \left(\vec{\nabla} \times \vec{\mathbf{F}}\right) \cdot \vec{\mathbf{G}} \;-\; \vec{\mathbf{F}} \cdot \left(\vec{\nabla} \times \vec{\mathbf{G}}\right)$$
div (**F** × **G**) = (curl **F**)•**G** − **F**•(curl **G**)

$$\vec{\nabla} \times \left(\vec{\mathbf{F}} \times \vec{\mathbf{G}}\right) \;=\; \left(\vec{\mathbf{G}} \cdot \vec{\nabla}\right)\vec{\mathbf{F}} \;-\; \left(\vec{\mathbf{F}} \cdot \vec{\nabla}\right)\vec{\mathbf{G}} \;+\; \left(\vec{\nabla} \cdot \vec{\mathbf{G}}\right)\vec{\mathbf{F}} \;-\; \left(\vec{\nabla} \cdot \vec{\mathbf{F}}\right)\vec{\mathbf{G}},$$

where
$$\left(\vec{\mathbf{F}} \cdot \vec{\nabla}\right)\vec{\mathbf{G}} \;=\; \left(F_1 \frac{\partial G_1}{\partial x} + F_2 \frac{\partial G_1}{\partial y} + F_3 \frac{\partial G_1}{\partial z}\right)\hat{\mathbf{i}} + \left(F_1 \frac{\partial G_2}{\partial x} + F_2 \frac{\partial G_2}{\partial y} + F_3 \frac{\partial G_2}{\partial z}\right)\hat{\mathbf{j}} + \left(F_1 \frac{\partial G_3}{\partial x} + F_2 \frac{\partial G_3}{\partial y} + F_3 \frac{\partial G_3}{\partial z}\right)\hat{\mathbf{k}}$$

so that $\left(\vec{\mathbf{F}} \cdot \vec{\nabla}\right)$ is the operator $\left(F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y} + F_3 \frac{\partial}{\partial z}\right)$

$$\vec{\nabla}\left(\vec{\mathbf{F}} \cdot \vec{\mathbf{G}}\right) \;=\; \left(\vec{\mathbf{G}} \cdot \vec{\nabla}\right)\vec{\mathbf{F}} \;+\; \left(\vec{\mathbf{F}} \cdot \vec{\nabla}\right)\vec{\mathbf{G}} \;+\; \vec{\mathbf{G}} \times \left(\vec{\nabla} \times \vec{\mathbf{F}}\right) \;+\; \vec{\mathbf{F}} \times \left(\vec{\nabla} \times \vec{\mathbf{G}}\right)$$

[End of Chapter 6]

[Space for Additional Notes]