7. Conversions between Coordinate Systems

In general, the conversion of a vector $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$ from Cartesian coordinates (x, y, z) to another orthonormal coordinate system (u, v, w) in $\mathbb{R}^3$ (where "orthonormal" means that the new basis vectors $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$ are mutually orthogonal and of unit length) is given by $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}} = F_u\hat{\mathbf{u}} + F_v\hat{\mathbf{v}} + F_w\hat{\mathbf{w}}$.

However, $F_u = \vec{\mathbf{F}}\cdot\hat{\mathbf{u}} = \left(F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}\right)\cdot\hat{\mathbf{u}} = \left(\hat{\mathbf{i}}\cdot\hat{\mathbf{u}}\right)F_x + \left(\hat{\mathbf{j}}\cdot\hat{\mathbf{u}}\right)F_y + \left(\hat{\mathbf{k}}\cdot\hat{\mathbf{u}}\right)F_z$.

F_v and F_w are defined similarly in terms of the Cartesian components F_x, F_y, F_z.

In matrix form

$$\begin{bmatrix} F_u \\ F_v \\ F_w \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}\cdot\hat{\mathbf{u}} & \hat{\mathbf{j}}\cdot\hat{\mathbf{u}} & \hat{\mathbf{k}}\cdot\hat{\mathbf{u}} \\ \hat{\mathbf{i}}\cdot\hat{\mathbf{v}} & \hat{\mathbf{j}}\cdot\hat{\mathbf{v}} & \hat{\mathbf{k}}\cdot\hat{\mathbf{v}} \\ \hat{\mathbf{i}}\cdot\hat{\mathbf{w}} & \hat{\mathbf{j}}\cdot\hat{\mathbf{w}} & \hat{\mathbf{k}}\cdot\hat{\mathbf{w}} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}.$$

The matrices on the right hand side of the equation will contain a mixture of expressions in the new (u, v, w) and old (x, y, z) coordinates. This needs to be converted into a set of expressions in (u, v, w) only.

Example 7.01

Express the vector $\vec{\mathbf{F}} = y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ in cylindrical polar coordinates.

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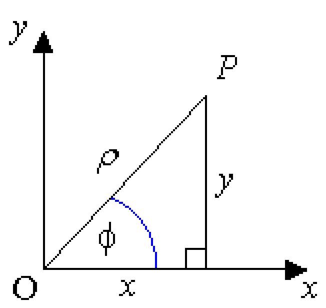
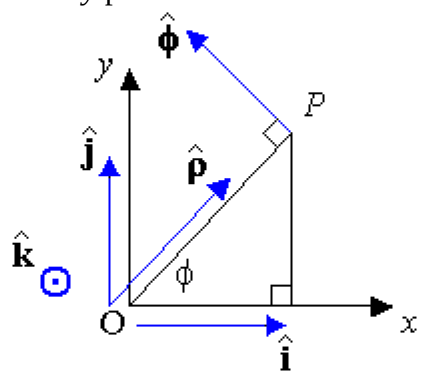
x y plane: coordinates



x y plane: basis vectors



$\Rightarrow \quad x =$ $\qquad \hat{\mathbf{i}}\cdot\hat{\rho} =$

$y =$ $\qquad \hat{\mathbf{j}}\cdot\hat{\rho} =$

$z =$ $\qquad \hat{\mathbf{k}}\cdot\hat{\rho} =$

Example 7.01 (continued)

$\hat{\mathbf{i}} \bullet \hat{\boldsymbol{\phi}} =$

$\hat{\mathbf{i}} \bullet \hat{\mathbf{k}} =$

$\hat{\mathbf{j}} \bullet \hat{\boldsymbol{\phi}} =$

$\hat{\mathbf{j}} \bullet \hat{\mathbf{k}} =$

$\hat{\mathbf{k}} \bullet \hat{\boldsymbol{\phi}} =$

$\hat{\mathbf{k}} \bullet \hat{\mathbf{k}} =$

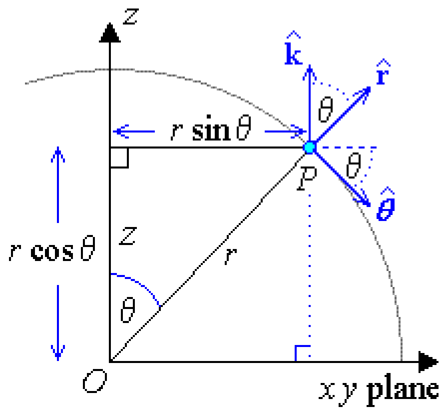
The coefficient conversion matrix from Cartesian to cylindrical polar is therefore

Letting $c \equiv \cos\phi, \quad s \equiv \sin\phi$:

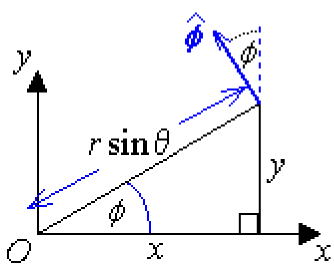
$\vec{\mathbf{F}}_{\text{polar}} =$

We can also generate the coordinate transformation matrix from Cartesian coordinates (x, y, z) to **spherical polar** coordinates (r, θ, ϕ).
[θ is the declination (angle down from the north pole, $0 \le \theta \le \pi$) and ϕ is the azimuth (angle around the equator $0 \le \phi < 2\pi$).]

[Vertical] Plane containing z-axis and radial vector $\vec{\mathbf{r}}$:



[Horizontal] Plane $z = r\cos\theta$:



The conversion matrix from Cartesian to spherical polar coordinates is then

$$\begin{bmatrix} \hat{\mathbf{i}} \bullet \hat{\mathbf{r}} & \hat{\mathbf{j}} \bullet \hat{\mathbf{r}} & \hat{\mathbf{k}} \bullet \hat{\mathbf{r}} \\ \hat{\mathbf{i}} \bullet \hat{\boldsymbol{\theta}} & \hat{\mathbf{j}} \bullet \hat{\boldsymbol{\theta}} & \hat{\mathbf{k}} \bullet \hat{\boldsymbol{\theta}} \\ \hat{\mathbf{i}} \bullet \hat{\boldsymbol{\phi}} & \hat{\mathbf{j}} \bullet \hat{\boldsymbol{\phi}} & \hat{\mathbf{k}} \bullet \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}$$

Example 7.02

Convert $\vec{\mathbf{F}} = y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}}$ to spherical polar coordinates.

Let $c_\theta \equiv \cos\theta, \quad s_\theta \equiv \sin\theta, \quad c_\phi \equiv \cos\phi, \quad s_\phi \equiv \sin\phi$

$$\vec{\mathbf{F}} = \begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix} =$$

Expressions for the gradient, divergence, curl and Laplacian operators in any orthonormal coordinate system will follow later in this chapter.

Summary for Coordinate Conversion:

To convert a vector expressed in Cartesian components $v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}$ into the equivalent vector expressed in **cylindrical polar** coordinates $v_\rho \hat{\boldsymbol{\rho}} + v_\phi \hat{\boldsymbol{\phi}} + v_z \hat{\mathbf{k}}$, express the Cartesian components v_x, v_y, v_z in terms of $(\rho,\ \phi,\ z)$ using
$x = \rho \cos\phi,\ \ y = \rho \sin\phi,\ \ z = z$; then evaluate

$$\begin{bmatrix} v_\rho \\ v_\phi \\ v_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Use the inverse matrix to transform back to Cartesian coordinates:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_\rho \\ v_\phi \\ v_z \end{bmatrix}$$

To convert a vector expressed in Cartesian components $v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}$ into the equivalent vector expressed in **spherical polar** coordinates $v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}} + v_\phi \hat{\boldsymbol{\phi}}$, express the Cartesian components v_x, v_y, v_z in terms of $(r,\ \theta,\ \phi)$ using
$x = r \sin\theta \cos\phi,\quad y = r \sin\theta \sin\phi,\quad z = r\cos\theta$; then evaluate

$$\begin{bmatrix} v_r \\ v_\theta \\ v_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Use the inverse matrix to transform back to Cartesian coordinates:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} v_r \\ v_\theta \\ v_\phi \end{bmatrix}$$

Note that, in both cases, the transformation matrix A is orthogonal, so that $A^{-1} = A^T$. This is not true for most square matrices A, but it is generally true for transformations between orthonormal coordinate systems.

Basis Vectors in Other Coordinate Systems

In the Cartesian coordinate system, all three basis vectors are absolute constants:

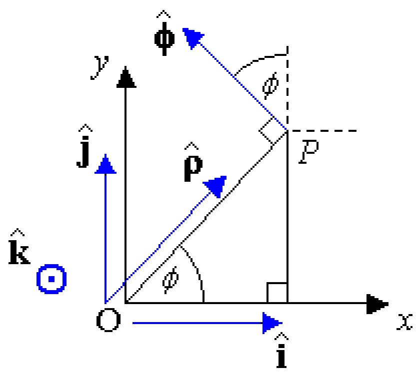
$$\frac{d}{dt}\hat{\mathbf{i}} = \frac{d}{dt}\hat{\mathbf{j}} = \frac{d}{dt}\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

The derivative of a vector is then straightforward to calculate:

$$\frac{d}{dt}\left(f_1\hat{\mathbf{i}} + f_2\hat{\mathbf{j}} + f_3\hat{\mathbf{k}}\right) = \hat{\mathbf{i}}\frac{df_1}{dt} + \hat{\mathbf{j}}\frac{df_2}{dt} + \hat{\mathbf{k}}\frac{df_3}{dt}$$

But most non-Cartesian basis vectors are not constant.

Cylindrical Polar:



$\hat{\boldsymbol{\rho}} =$

$\hat{\boldsymbol{\phi}} =$

$\hat{\mathbf{k}} = \hat{\mathbf{k}}$

Let $\dot{\mathbf{v}} \equiv \dfrac{d\vec{\mathbf{v}}}{dt}$ then $\dot{\hat{\boldsymbol{\rho}}} =$

$\dot{\hat{\boldsymbol{\phi}}} =$

$\dot{\hat{\mathbf{k}}} =$

Therefore if a vector $\vec{\mathbf{F}}$ is described in cylindrical polar coordinates
$\vec{\mathbf{F}} = F_\rho\hat{\boldsymbol{\rho}} + F_\phi\hat{\boldsymbol{\phi}} + F_z\hat{\mathbf{k}}$, then

$\dot{\vec{\mathbf{F}}} =$

In particular, the displacement vector is $\vec{\mathbf{r}}(t) = \rho(t)\hat{\boldsymbol{\rho}} + 0\hat{\boldsymbol{\phi}} + z(t)\hat{\mathbf{k}}$, so that the velocity vector is

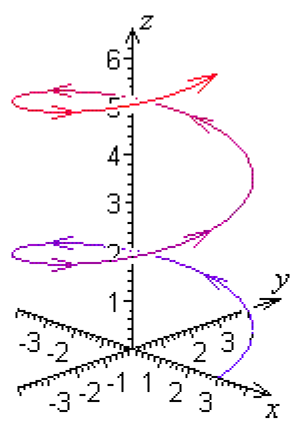
$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d\rho}{dt}\hat{\boldsymbol{\rho}} + \rho\frac{d\phi}{dt}\hat{\boldsymbol{\phi}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

Example 7.03

Find the velocity and acceleration in cylindrical polar coordinates for a particle travelling along the helix $x = 3\cos 2t$, $y = 3\sin 2t$, $z = t$.

Cylindrical polar coordinates: $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

$$\Rightarrow \quad \rho^2 = x^2 + y^2, \quad \tan\phi = \frac{y}{x}$$



Alternative derivation of cylindrical polar basis vectors

On page 7.02 we derived the coordinate conversion matrix A to convert a vector expressed in Cartesian components $v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}$ into the equivalent vector expressed in cylindrical polar coordinates $v_\rho \hat{\boldsymbol{\rho}} + v_\phi \hat{\boldsymbol{\phi}} + v_z \hat{\mathbf{k}}$

$$\begin{bmatrix} v_\rho \\ v_\phi \\ v_z \end{bmatrix} = \mathrm{A} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

and its inverse A^{-1}

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \mathrm{A}^{-1} \begin{bmatrix} v_\rho \\ v_\phi \\ v_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_\rho \\ v_\phi \\ v_z \end{bmatrix}$$

For *any* 3×3 matrix M,

$\mathrm{M}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =$ the first column of M; $\mathrm{M}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} =$ the second column of M; and

$\mathrm{M}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} =$ the third column of M

Note that $\hat{\boldsymbol{\rho}} = 1\hat{\boldsymbol{\rho}} + 0\hat{\boldsymbol{\phi}} + 0\hat{\mathbf{k}}$, so that the Cartesian form of $\hat{\boldsymbol{\rho}}$ is

$$\hat{\boldsymbol{\rho}} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}$$

Similarly

$$\hat{\boldsymbol{\phi}} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}$$

Therefore the columns of A^{-1} (and therefore the rows of A) are just the Cartesian components of the three cylindrical polar basis vectors.

Spherical Polar Coordinates

The coordinate conversion matrix also provides a quick route to finding the Cartesian components of the three basis vectors of the spherical polar coordinate system.

$$\hat{\mathbf{r}} = 1\hat{\mathbf{r}} + 0\hat{\boldsymbol{\theta}} + 0\hat{\boldsymbol{\phi}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\text{sph}}$$

$$\hat{\mathbf{r}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \mathrm{A}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$$

$$\Rightarrow \quad \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}$$

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \mathrm{A}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{bmatrix}$$

$$\Rightarrow \quad \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}$$

$$\hat{\boldsymbol{\phi}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{bmatrix} = \mathrm{A}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}$$

Again, the Cartesian components of the basis vectors are just the columns of A^{-1} (which are also the rows of A). This is true for any orthonormal coordinate system.

These expressions for $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ can also be found geometrically
[available on the course web site].

Derivatives of the Spherical Polar Basis Vectors

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}$$

$$\Rightarrow \quad \frac{d\hat{\mathbf{r}}}{dt} =$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}$$

$$\Rightarrow \quad \frac{d\hat{\boldsymbol{\theta}}}{dt} =$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}$$

$$\Rightarrow \quad \frac{d\hat{\boldsymbol{\phi}}}{dt} =$$

In particular, the displacement vector is $\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$, so that the velocity vector is

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\left(\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \sin\theta\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}\right)$$

$$\boxed{\Rightarrow \quad \vec{\mathbf{v}} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\sin\theta\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}}$$

It can be shown that the acceleration vector in the spherical polar coordinate system is

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \left(\frac{d^2 r}{dt^2} - r\left(\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\phi}{dt}\right)^2 \sin^2\theta \right) \right)\hat{\mathbf{r}}$$

$$+ \left(\frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt} \right) - \frac{r}{2}\left(\frac{d\phi}{dt}\right)^2 \sin 2\theta \right)\hat{\boldsymbol{\theta}}$$

$$+ \left(\frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2 \frac{d\phi}{dt}\sin^2\theta \right) \right)\hat{\boldsymbol{\phi}}$$

Compare this to the Cartesian equivalent $\vec{\mathbf{a}} = \frac{d^2 x}{dt^2}\hat{\mathbf{i}} + \frac{d^2 y}{dt^2}\hat{\mathbf{j}} + \frac{d^2 z}{dt^2}\hat{\mathbf{k}}$!

Example 7.04

Find the velocity vector $\vec{\mathbf{v}}$ for a particle whose displacement vector $\vec{\mathbf{r}}$, in spherical polar coordinates, is given by $r = 4,\ \theta = t,\ \phi = 2t,\quad (0 < t < \pi)$.

$$r = 4, \quad \theta = t, \quad \phi = 2t \quad \Rightarrow \quad \frac{dr}{dt} = 0, \quad \frac{d\theta}{dt} = 1, \quad \frac{d\phi}{dt} = 2$$

$$\vec{\mathbf{v}} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\sin\theta\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

<u>Summary</u>:

<u>Cylindrical Polar</u>:

$$\frac{d}{dt}\hat{\boldsymbol{\rho}} = \frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

$$\frac{d}{dt}\hat{\boldsymbol{\phi}} = -\frac{d\phi}{dt}\hat{\boldsymbol{\rho}}$$

$$\frac{d}{dt}\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

$$\vec{\mathbf{r}} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{k}} \quad \Rightarrow \quad \vec{\mathbf{v}} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\mathbf{k}}$$

<u>Spherical Polar</u>:

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \sin\theta\,\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{r}} + \cos\theta\,\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

$$\frac{d\hat{\boldsymbol{\phi}}}{dt} = -\left(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}\right)\frac{d\phi}{dt}$$

$$\vec{\mathbf{r}} = r\,\hat{\mathbf{r}} \quad \Rightarrow \quad \vec{\mathbf{v}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}}$$

Gradient Operator in Other Coordinate Systems

For any orthogonal curvilinear coordinate system (u_1, u_2, u_3) in $\mathbb{R}^3$,

a tangent vector along the u_i curvilinear axis is $\vec{\mathbf{T}}_i = \dfrac{\partial \vec{\mathbf{r}}}{\partial u_i}$

The unit tangent vectors along the curvilinear axes are $\hat{\mathbf{e}}_i = \mathbf{T}_i = \dfrac{1}{h_i}\dfrac{\partial \vec{\mathbf{r}}}{\partial u_i}$,

where the scale factors $h_i = \left|\dfrac{\partial \vec{\mathbf{r}}}{\partial u_i}\right|$.

The displacement vector $\vec{\mathbf{r}}$ can then be written as $\vec{\mathbf{r}} = u_1\hat{\mathbf{e}}_1 + u_2\hat{\mathbf{e}}_2 + u_3\hat{\mathbf{e}}_3$, where the unit vectors $\hat{\mathbf{e}}_i$ form an **orthonormal basis** for $\mathbb{R}^3$:

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$$

[δ_{ij} is the "Kronecker delta".]

The differential displacement vector $\mathbf{d}\vec{\mathbf{r}}$ is (by the Chain Rule)

$$\mathbf{d}\vec{\mathbf{r}} = \frac{\partial \vec{\mathbf{r}}}{\partial u_1} du_1 + \frac{\partial \vec{\mathbf{r}}}{\partial u_2} du_2 + \frac{\partial \vec{\mathbf{r}}}{\partial u_3} du_3 = h_1\, du_1\, \hat{\mathbf{e}}_1 + h_2\, du_2\, \hat{\mathbf{e}}_2 + h_3\, du_3\, \hat{\mathbf{e}}_3$$

which leads to a more general expression for the velocity vector (compared to those of the preceding page):

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = h_1 \frac{du_1}{dt}\hat{\mathbf{e}}_1 + h_2 \frac{du_2}{dt}\hat{\mathbf{e}}_2 + h_3 \frac{du_3}{dt}\hat{\mathbf{e}}_3$$

The differential arc length ds is

$$ds^2 = \mathbf{d}\vec{\mathbf{r}} \bullet \mathbf{d}\vec{\mathbf{r}} = (h_1\, du_1)^2 + (h_2\, du_2)^2 + (h_3\, du_3)^2$$

The element of volume dV is

$$dV = h_1 h_2 h_3\, du_1 du_2 du_3 = \left|\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}\right| du_1 du_2 du_3$$

$$= \begin{vmatrix} \dfrac{\partial x}{\partial u_1} & \dfrac{\partial y}{\partial u_1} & \dfrac{\partial z}{\partial u_1} \\ \dfrac{\partial x}{\partial u_2} & \dfrac{\partial y}{\partial u_2} & \dfrac{\partial z}{\partial u_2} \\ \dfrac{\partial x}{\partial u_3} & \dfrac{\partial y}{\partial u_3} & \dfrac{\partial z}{\partial u_3} \end{vmatrix} du_1 du_2 du_3$$

Gradient operator $$\vec{\nabla} = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial}{\partial u_3}$$

Gradient $$\vec{\nabla} V = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial V}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial V}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial V}{\partial u_3}$$

Divergence $$\vec{\nabla} \bullet \vec{\mathbf{F}} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_2 h_3 f_1)}{\partial u_1} + \frac{\partial(h_3 h_1 f_2)}{\partial u_2} + \frac{\partial(h_1 h_2 f_3)}{\partial u_3}\right)$$

Curl $$\vec{\nabla} \times \vec{\mathbf{F}} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & \dfrac{\partial}{\partial u_1} & h_1 f_1 \\ h_2 \hat{\mathbf{e}}_2 & \dfrac{\partial}{\partial u_2} & h_2 f_2 \\ h_3 \hat{\mathbf{e}}_3 & \dfrac{\partial}{\partial u_3} & h_3 f_3 \end{vmatrix}$$

Laplacian $$\nabla^2 V = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial u_3}\right)\right)$$

Cartesian: $h_x = h_y = h_z = 1$

Cylindrical polar: $h_\rho = h_z = 1, \quad h_\phi = \rho$

Spherical polar: $h_r = 1, \quad h_\theta = r, \quad h_\phi = r \sin\theta$

The familiar expressions then follow for the Cartesian coordinate system.

In **cylindrical polar coordinates**, naming the three basis vectors as $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{k}},$ we have:

$$\vec{\mathbf{r}} = \rho \hat{\boldsymbol{\rho}} + 0 \hat{\boldsymbol{\phi}} + z \hat{\mathbf{k}} = \begin{bmatrix} \rho & 0 & z \end{bmatrix}^{\mathrm{T}}$$

The relationship to the Cartesian coordinate system is

$$x = \rho \cos \phi , \quad y = \rho \sin \phi , \quad z = z \quad \Rightarrow \quad \rho^2 = x^2 + y^2 , \quad \tan \phi = \frac{y}{x}$$

One scale factor is

$$h_\rho = \left| \frac{\partial \vec{\mathbf{r}}}{\partial \rho} \right| =$$

In a similar way, we can confirm that $h_\phi = \rho$ and $h_z = 1$.

In cylindrical polar coordinates,

$$dV =$$

$$ds^2 =$$

$$\vec{\nabla} V =$$

$$\vec{\nabla} \bullet \vec{\mathbf{F}} =$$

$$\vec{\nabla} \times \vec{\mathbf{F}} =$$

$$\nabla^2 V =$$

All of the above are undefined on the z-axis ($\rho = 0$), where there is a coordinate singularity. However, by taking the limit as $\rho \to 0$, we may obtain well-defined values for some or all of the above expressions.

Example 7.05

Given that the gradient operator in a general curvilinear coordinate system is

$\vec{\nabla} = \left(\frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial}{\partial u_3} \right)$, why isn't the divergence of

$\vec{\mathbf{F}} = F_1\hat{\mathbf{e}}_1 + F_2\hat{\mathbf{e}}_2 + F_3\hat{\mathbf{e}}_3$ equal, in general, to $\left(\frac{1}{h_1}\frac{\partial F_1}{\partial u_1} + \frac{1}{h_2}\frac{\partial F_2}{\partial u_2} + \frac{1}{h_3}\frac{\partial F_3}{\partial u_3} \right)$?

The quick answer is that the differential operators operate not just on the components F_1, F_2, F_3, but also on the basis vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$. In most orthonormal coordinate systems, these basis vectors are not constant. The divergence therefore contains additional terms.

$$\left(\frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial u_3} \right) \bullet \left(F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3 \right) =$$

$$\left(\frac{\mathbf{e}_1 \bullet \mathbf{e}_1}{h_1}\frac{\partial F_1}{\partial u_1} + \frac{F_1}{h_1}\mathbf{e}_1 \bullet \frac{\partial \mathbf{e}_1}{\partial u_1} \right) + \left(\frac{\mathbf{e}_2 \bullet \mathbf{e}_1}{h_2}\frac{\partial F_1}{\partial u_2} + \frac{F_1}{h_2}\mathbf{e}_2 \bullet \frac{\partial \mathbf{e}_1}{\partial u_2} \right) + \left(\frac{\mathbf{e}_3 \bullet \mathbf{e}_1}{h_3}\frac{\partial F_1}{\partial u_3} + \frac{F_1}{h_3}\mathbf{e}_3 \bullet \frac{\partial \mathbf{e}_1}{\partial u_3} \right) +$$
$$\left(\frac{\mathbf{e}_1 \bullet \mathbf{e}_2}{h_1}\frac{\partial F_2}{\partial u_1} + \frac{F_2}{h_1}\mathbf{e}_1 \bullet \frac{\partial \mathbf{e}_2}{\partial u_1} \right) + \left(\frac{\mathbf{e}_2 \bullet \mathbf{e}_2}{h_2}\frac{\partial F_2}{\partial u_2} + \frac{F_2}{h_2}\mathbf{e}_2 \bullet \frac{\partial \mathbf{e}_2}{\partial u_2} \right) + \left(\frac{\mathbf{e}_3 \bullet \mathbf{e}_2}{h_3}\frac{\partial F_2}{\partial u_3} + \frac{F_2}{h_3}\mathbf{e}_3 \bullet \frac{\partial \mathbf{e}_2}{\partial u_3} \right) +$$
$$\left(\frac{\mathbf{e}_1 \bullet \mathbf{e}_3}{h_1}\frac{\partial F_3}{\partial u_1} + \frac{F_3}{h_1}\mathbf{e}_1 \bullet \frac{\partial \mathbf{e}_3}{\partial u_1} \right) + \left(\frac{\mathbf{e}_2 \bullet \mathbf{e}_3}{h_2}\frac{\partial F_3}{\partial u_2} + \frac{F_3}{h_2}\mathbf{e}_2 \bullet \frac{\partial \mathbf{e}_3}{\partial u_2} \right) + \left(\frac{\mathbf{e}_3 \bullet \mathbf{e}_3}{h_3}\frac{\partial F_3}{\partial u_3} + \frac{F_3}{h_3}\mathbf{e}_3 \bullet \frac{\partial \mathbf{e}_3}{\partial u_3} \right) =$$

$$\left(\frac{1}{h_1}\frac{\partial F_1}{\partial u_1} + \frac{F_1}{h_1}\mathbf{e}_1 \bullet \frac{\partial \mathbf{e}_1}{\partial u_1} \right) + \left(\frac{F_1}{h_2}\mathbf{e}_2 \bullet \frac{\partial \mathbf{e}_1}{\partial u_2} \right) + \left(\frac{F_1}{h_3}\mathbf{e}_3 \bullet \frac{\partial \mathbf{e}_1}{\partial u_3} \right) +$$
$$\left(\frac{F_2}{h_1}\mathbf{e}_1 \bullet \frac{\partial \mathbf{e}_2}{\partial u_1} \right) + \left(\frac{1}{h_2}\frac{\partial F_2}{\partial u_2} + \frac{F_2}{h_2}\mathbf{e}_2 \bullet \frac{\partial \mathbf{e}_2}{\partial u_2} \right) + \left(\frac{F_2}{h_3}\mathbf{e}_3 \bullet \frac{\partial \mathbf{e}_2}{\partial u_3} \right) +$$
$$\left(\frac{F_3}{h_1}\mathbf{e}_1 \bullet \frac{\partial \mathbf{e}_3}{\partial u_1} \right) + \left(\frac{F_3}{h_2}\mathbf{e}_2 \bullet \frac{\partial \mathbf{e}_3}{\partial u_2} \right) + \left(\frac{1}{h_3}\frac{\partial F_3}{\partial u_3} + \frac{F_3}{h_3}\mathbf{e}_3 \bullet \frac{\partial \mathbf{e}_3}{\partial u_3} \right)$$

For Cartesian coordinates, all derivatives of any basis vector are zero, which leaves the familiar Cartesian expression for the divergence. But for most non-Cartesian coordinate systems, at least some of these partial derivatives are not zero. More complicated expressions for the divergence therefore arise.

Example 7.05 (continued)

For **cylindrical polar coordinates**, we have

$$\left(\frac{1}{1}\frac{\partial F_\rho}{\partial \rho} + \frac{F_\rho}{1}\hat{\boldsymbol{\rho}} \bullet \frac{\partial \hat{\boldsymbol{\rho}}}{\partial \rho}\right) + \left(\frac{F_\rho}{\rho}\hat{\boldsymbol{\phi}} \bullet \frac{\partial \hat{\boldsymbol{\rho}}}{\partial \phi}\right) + \left(\frac{F_\rho}{1}\hat{\mathbf{k}} \bullet \frac{\partial \hat{\boldsymbol{\rho}}}{\partial z}\right) +$$

$$\left(\frac{F_\phi}{1}\hat{\boldsymbol{\rho}} \bullet \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \rho}\right) + \left(\frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{F_\phi}{\rho}\hat{\boldsymbol{\phi}} \bullet \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi}\right) + \left(\frac{F_\phi}{1}\hat{\mathbf{k}} \bullet \frac{\partial \hat{\boldsymbol{\phi}}}{\partial z}\right) +$$

$$\left(\frac{F_z}{1}\hat{\boldsymbol{\rho}} \bullet \frac{\partial \hat{\mathbf{k}}}{\partial \rho}\right) + \left(\frac{F_z}{\rho}\hat{\boldsymbol{\phi}} \bullet \frac{\partial \hat{\mathbf{k}}}{\partial \phi}\right) + \left(\frac{1}{1}\frac{\partial F_z}{\partial z} + \frac{F_z}{1}\hat{\mathbf{k}} \bullet \frac{\partial \hat{\mathbf{k}}}{\partial z}\right)$$

But none of the basis vectors varies with ρ or z and the basis vector $\hat{\mathbf{k}}$ is absolutely constant. Therefore the divergence becomes

$$\left(\frac{1}{1}\frac{\partial F_\rho}{\partial \rho} + 0\right) + \left(\frac{F_\rho}{\rho}\hat{\boldsymbol{\phi}} \bullet \frac{\partial \hat{\boldsymbol{\rho}}}{\partial \phi}\right) + (0) +$$

$$(0) + \left(\frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{F_\phi}{\rho}\hat{\boldsymbol{\phi}} \bullet \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi}\right) + (0) + \ (0) + (0) + \left(\frac{1}{1}\frac{\partial F_z}{\partial z} + 0\right)$$

But $\dfrac{\partial \hat{\boldsymbol{\rho}}}{\partial \phi} = \hat{\boldsymbol{\phi}} \;\Rightarrow\; \left(\dfrac{F_\rho}{\rho}\hat{\boldsymbol{\phi}} \bullet \dfrac{\partial \hat{\boldsymbol{\rho}}}{\partial \phi}\right) = \left(\dfrac{F_\rho}{\rho}\hat{\boldsymbol{\phi}} \bullet \hat{\boldsymbol{\phi}}\right) = \dfrac{F_\rho}{\rho}$

and $\dfrac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} = -\hat{\boldsymbol{\rho}} \;\Rightarrow\; \left(\dfrac{F_\phi}{\rho}\hat{\boldsymbol{\phi}} \bullet \dfrac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi}\right) = \left(\dfrac{F_\phi}{\rho}\hat{\boldsymbol{\phi}} \bullet (-\hat{\boldsymbol{\rho}})\right) = 0$

So we recover the cylindrical polar form for the divergence,

$$\operatorname{div}\vec{\mathbf{F}} = \frac{\partial F_\rho}{\partial \rho} + \frac{F_\rho}{\rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

In **spherical polar coordinates**, naming the three basis vectors as $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}},$ we have:

$\vec{\mathbf{r}} = r\hat{\mathbf{r}} + 0\hat{\boldsymbol{\theta}} + 0\hat{\boldsymbol{\phi}} = \left[r \;\; 0 \;\; 0 \right]^{\mathrm{T}}$

The relationship to the Cartesian coordinate system is

$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta .$

One of the scale factors is

$$h_\theta = \left| \frac{\partial \vec{\mathbf{r}}}{\partial \theta} \right| =$$

In a similar way, we can confirm that $h_r = 1$ and $h_\phi = r\sin\theta$.

$$dV =$$

$$ds^2 =$$

$$\vec{\boldsymbol{\nabla}} V =$$

$$\vec{\boldsymbol{\nabla}} \bullet \vec{\mathbf{F}} =$$

Spherical Polar (continued)

$\vec{\nabla} \times \vec{\mathbf{F}} =$

$\nabla^2 V =$

All of the above are undefined on the z-axis ($\sin\theta = 0$), where there is a coordinate singularity. However, by taking the limit as $\sin\theta \to 0$, we may obtain well-defined values for some or all of the above expressions. The origin ($r = 0$) poses a similar problem.

Example 7.06

A vector field has the equation, in cylindrical polar coordinates $(\rho,\ \phi,\ z)$,

$$\vec{\mathbf{F}} = \frac{k}{\rho^n}\hat{\mathbf{e}}_\rho = \frac{k}{\rho^n}\hat{\boldsymbol{\rho}}$$

Find the divergence of $\vec{\mathbf{F}}$ and the value of n for which the divergence vanishes for all $\rho > 0$.

Example 7.07

In spherical polar coordinates,

$$\vec{\mathbf{F}}(r,\theta,\phi) = f(\phi)\cot\theta\,\hat{\mathbf{r}} - 2f(\phi)\,\hat{\boldsymbol{\theta}} + g(r,\theta)\,\hat{\boldsymbol{\phi}} ,$$

where $f(\phi)$ is *any* differentiable function of ϕ only
and $g(r,\theta)$ is *any* differentiable function of r and θ only.

Find the divergence of $\vec{\mathbf{F}}$.

Example 7.08

Find $\operatorname{curl}\left(\sin\theta\left(\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}\right)\right)$, where θ, ϕ are the two angular coordinates in the standard spherical polar coordinate system.

Central Force Law

If a potential function $V(x, y, z)$, (due solely to a point source at the origin) depends only on the distance r from the origin, then the functional form of the potential can be deduced. Using spherical polar coordinates:

$$V(r, \theta, \phi) = f(r)$$

$$\Rightarrow \quad \nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{df}{dr}\right) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$$

But, in any regions not containing any sources of the vector field, the divergence of the vector field $\vec{\mathbf{F}} = \vec{\boldsymbol{\nabla}} V$ (and therefore the Laplacian of the associated potential function V) must be zero. Therefore, for all $r \neq 0$, Gravity is an example of a central force law, for which the potential function must be of the form $V(r,\theta,\phi) = A - \frac{B}{r}$. The zero point for the potential is often set at infinity:

$$\lim_{r\to\infty} V = \lim_{r\to\infty}\left(A - \frac{B}{r}\right) = A = 0$$

The force per unit mass due to gravity from a point mass M at the origin is

$$\vec{\mathbf{F}} = -\vec{\mathbf{\nabla}}V = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

But, in spherical polar coordinates,

$$\vec{\mathbf{\nabla}}V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial V}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial V}{\partial \phi} = \hat{\mathbf{r}}\frac{dV}{dr} = \hat{\mathbf{r}}\frac{B}{r^2}$$

$$\Rightarrow \quad -\frac{GM}{r^2} = -\frac{B}{r^2} \quad \Rightarrow \quad B = GM$$

Therefore the gravitational potential function is

$$V(r) = -\frac{GM}{r}$$

The electrostatic potential function is similar, with a different constant of proportionality.

[Space for additional notes]

[End of Chapter 7]